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A Newmark-Type Integrator for Flexible Systems Considering Nonsmooth Unilateral Constraints

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ABSTRACT

Mechanical systems are usually subjected not only to bilateral constraints but also to unilateral constraints. Inspired by the HHT time integration method for smooth flexible multibody dynamics, this paper presents a Newmark-type integrator, denoted by nonsmooth HHT method, to include the nonsmooth property of unilateral constraints. Through numerical examples accounting for both rigid and flexible body models, a bunch of methods are compared with the unilateral constraints on both velocity level and position level. Results show that the presented nonsmooth HHT method benefits from the accuracy and stability property of the classical HHT method with controllable numerical damping. In particular, when it comes to the analysis of flexible systems, the nonsmooth HHT method shows much better accuracy property than that of the other methods, including the Moreau time-stepping method and the fully implicit Newmark methods.

1 INTRODUCTION

Dynamics equations for a mechanical system with bilateral constraints can be characterized by a mixed set of second order differential and algebraic equations (DAEs). One of the classical research interests is in the efficient numerical solutions. Among the classical numerical solvers, the Hilber- Hughes-Taylor (HHT) method [1] and the generalized-α method [2], which are extended from the original implicit Newmark approach, are popular and have been extensively applied in structural dynamics. With an optimal choice of parameters, these solvers yield second-order accuracy and unconditional stability, which are of utmost important since the structural dynamics equations can be very stiff with a broad range of system eigenfrequencies [3]. Cardona and Géradin [4] proposed to apply these solvers to the index-3 DAEs in flexible multibody dynamics. These authors have demonstrated that, in the linear regime, the numerical instability in the Lagrangian multipliers can be removed with a small amount of numerical dissipation at high frequencies [3]. An extended analysis on the global convergence can be found in [5].

However, in reality, mechanical systems are usually subjected not only to the bilateral constraints but also to the unilateral constraints. For instance, the global dynamic behaviour of wind turbines is characterized by vibrations of structural elements such as the blades and the tower coupled with strong nonlinearities in the transmission line due to contacts between stiff or rigid bodies with backlash and friction phenomena. Unilateral constraints can be modelled either by regularization techniques, or by set-valued force laws that lead to nonsmooth models. Regularization techniques aim at replacing the discontinuity by a stiff but compliant model, see for example the continuous impact models proposed in [6]. However, the knowledge of a number of physical contact parameters such as contact stiffness and restitution coefficients, which are difficult to be obtained by experiments, can be required [6], and these techniques result in stiff differential equations and inexact solutions [7]. Nonsmooth models, represented as differential inclusions, complementarity systems or variational inequalities [8], are used in order to model the unilateral nature of the constraints, usually modeled by the Signorini condition. Nonsmooth models can also express impact laws, required by the unilateral constraints in finite–dimensional systems (rigid models or space–discretized systems). In the sequel, Newton's impact law will be used. When nonsmooth models are involved, one of the main computational challenges remains the numerical time integration of the dynamics of systems which may also be nonsmooth with impacts and jumps in velocities.

In the context of rigid multi-body systems, occurrences of impacts and/or abrupt changes in the velocities due to Signorini's condition and Coulomb's friction require great cares in the time integration method if we want to ensure consistency and stability. There exist two main groups of numerical integration schemes for nonsmooth systems, namely, event-driven schemes and time-stepping schemes. Event-driven schemes are based on an accurate event detection and the time step is adapted such that the end of step coincides with an event. At this time instant, the event is solved with the help of an impact law. Such schemes are accurate for the free flight smooth motions, but fail to handle frequent transitions in a short time and have no convergence proof [9]. In practice, they are only suitable for small multi-body systems with a few number of events. Contrary to the event–driven schemes, time-stepping schemes do not adapt their time step size on events but only on some accuracy requirements if needed. Two main families of schemes have been designed up to now : Schatzman–Paoli scheme [10, 11] based on central difference scheme and Moreau-Jean scheme [12, 13, 14] based on a θ-method. Time-stepping methods have been proven to be convergent and robust, and are extensively applied as the solution to the nonsmooth system models. In contrast to event-driven schemes, time-stepping schemes are expected to have order-one accuracy even in the smooth part of the motion [15, 16]. Thus, the accuracy is less satisfactory unless a very small step size is applied. However, it remains robust and efficient even for a large number of events as it is usual in structural dynamics. Attempts have recently been made to improve the global order of accuracy of time–stepping with nonsmooth events [17, 15, 18]. But for the moment, the application to nonsmooth mechanical systems of the most favorite time-stepping schemes for structural and flexible multi-body systems, i.e., HHT method and generalized-α method has not been successfully carried out. Within the time-stepping strategy, it remains difficult to expect to increase the order of accuracy of the scheme when an impact occurs. Nevertheless, the goal of the paper is to bridge this gap in order to benefit from the qualitative properties of the HHT and generalized-α schemes in terms of stability and controlled numerical damping.

In the context of computational contact mechanics of solids and structures, a lot of variants of the Newmark scheme have been proposed to solve the contact dynamics problems, see [19, 20] for two recent reviews. In these proposed approaches, it is implicitly assumed that the solutions including positions, displacements, velocities and contact forces are sufficiently smooth so that Newmark schemes of order 2 are applicable. Nevertheless, the lack of robustness and the numerical instabilities of the midpoint, trapezoidal rule and even dissipative HHT schemes, often leading to the blow-up of the numerical computation, have motivated the development of alternatives and modifications to standard integration schemes. To cite a few of them, the invocation of energy-conserving and dissipative schemes was the first attempt to remedy this problem. It generally results in a scheme of lower order where the contact forces or impulses are treated in a fully implicit manner [21, 22] and where the constraint are expressed at the velocity level [23, 24]. It should be noted that these requirements exactly meet the principles of design of the Moreau–Jean Time-stepping schemes [12, 13, 14], but in the latter case they are motivated by the nonsmoothness of solutions. By the way, the reader will find in [25] a comparison between the Moreau-Jean scheme and the Newmark scheme for space–discretized impacting bars. The convergence of the Newmark scheme with central differences and with an implicit treatment of the contact constraint at the position level has been proved in [26] for vibrating beams between two stops. Fortunately enough, it has been shown in [27] that the question of the coefficient of restitution is not raised on this case. The implicit treatment of the contact forces was also proposed in the context explicit integrators in a seminal paper [28]. Other attempts have been based on a) a mass redistribution which consists in removing the mass from the contact boundaries [29, 30] or b) a contact stabilization of the relative velocities at contact [31]. In our case, the contact activation between finite–freedom mechanical models induces the nonsmoothness of the solutions and, since rigid bodies are involved, it is difficult to remove masses and to stabilize contact without resorting to an impact law. Therefore, the direct application of higher order schemes in this context may be hazardous.

Within the framework structural dynamics with rigid and flexible bodies, this paper proposes to take the best the HHT method and the Moreau–Jean scheme by

- keeping the integration rule of the HHT scheme for the nonlinear smooth dynamics; the aim is to enjoy the stability, the robustness and the controlled numerical dissipation at high frequencies;

- dealing, as in the Moreau–Jean scheme, with the contact forces and impacts through their associated impulses; in this way we ensure that the evaluation of the contact reactions will be consistent when the times–step vanishes and an impact occurs;
- treating, as in the Moreau–Jean scheme, the unilateral constraint at the velocity level together with the impact law; this yields an energetically consistent treatment of the jumps in velocity together with a correct stabilization of the velocity at contact.

Even though the global order of accuracy can be brought back to order one due to the nonsmoothness [7, 8], second-order accuracy can still be achieved if no unilateral contacts occur. Thus, it aims at providing a general solver for structural dynamics, which can take both smooth and nonsmooth mechanics into account with robust order of accuracy. Especially, it will improve the control of high-frequency numerical damping in comparison with the Moreau–Jean time–stepping scheme based on the θ-method. Stability and convergence will be empirically analyzed and the properties of the proposed method will be illustrated based on academic examples.

2 Equations of motion

The equations of a flexible multibody system including bilateral and unilateral constraints can be expressed in the following form:

$$\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{v} = \mathbf{h}\,\mathrm{d}t + \mathbf{g}_{\mathbf{q}}^T \mathrm{d}\mathbf{i} \tag{1}$$

$$\mathbf{\Phi}(\mathbf{q}) = \mathbf{0} \tag{2}$$

$$\mathbf{0} \leq \mathbf{g}(\mathbf{q}) \perp \boldsymbol{\lambda}_u \geq \mathbf{0} \tag{3}$$

where

- $\mathbf{q}$ is the vector of generalized coordinates, e.g. absolute nodal coordinates;
- $\mathbf{h} = \mathbf{f}_{ext} - \mathbf{f}^{damp}(\mathbf{q}, \mathbf{v}) - \mathbf{f}^{int}(\mathbf{q}) - \mathbf{\Phi}_{\mathbf{q}}^T \boldsymbol{\lambda}_b$ collects the external, damping, internal and bilateral forces.
- $\mathbf{M}$ is the mass matrix;
- $\mathbf{\Phi}$ is the vector of bilateral holonomic constraints, $\mathbf{\Phi}_{\mathbf{q}}$ is the matrix of constraint gradients and $\boldsymbol{\lambda}_b$ is the vector of associated Lagrange multipliers;
- $\mathbf{g}$ is the vector of unilateral holonomic constraints and $\boldsymbol{\lambda}_u$ is the vector of associated Lagrange multipliers;
- $\mathrm{d}\mathbf{v}$ is the differential measure associated with the velocity $\mathbf{v}$ assumed to be of bounded variation;
- $\mathrm{d}t$ is the standard Lebesgue measure;
- $\mathrm{d}\mathbf{i}$ is the impulse measure of the contact reaction.

The complementarity condition $\mathbf{0} \leq \mathbf{g}(\mathbf{q}) \perp \boldsymbol{\lambda}_u \geq \mathbf{0}$ is expressed at the velocity level together with he Newton impact law as

$$\text{if } \mathbf{g}(\mathbf{q}) \leq \mathbf{0} \text{ then } \mathbf{0} \leq \mathbf{g}_{\mathbf{q}}\mathbf{v}(t^+) + e\,\mathbf{g}_{\mathbf{q}}\mathbf{v}(t^-) \perp \mathrm{d}\mathbf{i} \geq \mathbf{0} \tag{4}$$

where e is the coefficient of restitution.

Since the motion might be nonsmooth, jumps in the velocity and corresponding impacts are expected yielding to the following decomposition of the measures[1]:

$$\mathrm{d}\mathbf{v} = \dot{\mathbf{v}}\,\mathrm{d}t + \sum_i (\mathbf{v}(t_i^+) - \mathbf{v}(t_i^-))\,\boldsymbol{\delta}_{t_i} \tag{5}$$

$$\mathrm{d}\mathbf{i} = \boldsymbol{\lambda}_u\,\mathrm{d}t + \sum_i \mathbf{p}_i\,\boldsymbol{\delta}_{t_i} \tag{6}$$

[1] we neglect singular measures

where $(\mathbf{v}(t_i^+) - \mathbf{v}(t_i^-))$ is the jump in velocity at the instant t_i, δ_{t_i} the Dirac atom supported at t_i and p_i the impulse. Inserting (5) and (6) in (1) yield the standard equation of motion

$$\mathbf{M}\dot{\mathbf{v}}\mathrm{d}t = \mathbf{h}\,\mathrm{d}t + \mathbf{g}_{\mathbf{q}}^T\boldsymbol{\lambda}_u\mathrm{d}t \tag{7}$$

and the impact equation at time t_i

$$\mathbf{M}(\mathbf{v}(t_i^+) - \mathbf{v}(t_i^-)) = \mathbf{p}_i \tag{8}$$

3 Time integration method

In order to present the time stepping integration method, we propose to split the equation in a slightly different way by separating the contact part such that

$$\mathbf{M}\,\mathrm{d}\mathbf{v} = \mathbf{M}\mathbf{a}\,\mathrm{d}t + \mathbf{M}\,\mathrm{d}\mathbf{v}^{\mathrm{con}} \tag{9}$$

with

$$\begin{cases}\mathbf{M}\mathbf{a}\,\mathrm{d}t = \mathbf{h}\,\mathrm{d}t \\ \mathbf{M}\,\mathrm{d}\mathbf{v}^{\mathrm{con}} = \mathbf{g}_{\mathbf{q}}^T\mathrm{d}\mathbf{i}\end{cases} \tag{10}$$

where $\mathbf{a}$ is the acceleration and $\mathrm{d}\mathbf{v}^{\mathrm{con}}$ is the contribution to the velocity increments due to the contact impulse. As usual in the HHT scheme, the discrete evaluation of the first equation in (10) will be denoted as

$$\mathbf{M}\mathbf{a}_{n+1} = \mathbf{h}_{n+1} \tag{11}$$

For the impulsive part, we will keep the value of the impulse over the time–step as a primary unknown that is

$$\mathbf{M}\Delta\mathbf{v}_{n+1}^{\mathrm{con}} = \mathbf{g}_{\mathbf{q}}^T\boldsymbol{\Lambda}_{n+1} \tag{12}$$

The last relation (12) is a first-order approximation of the integral of the second equation in (10), that is

$$\int_{[t_n,t_{n+1})}\mathbf{M}\,\mathrm{d}\mathbf{v}^{\mathrm{con}} = \int_{[t_n,t_{n+1})}\mathbf{g}_{\mathbf{q}}^T\mathrm{d}\mathbf{i} \tag{13}$$

The discretization of the contact law is written as

$$\text{if } \mathbf{g}_n \leq \mathbf{0} \text{ then } \mathbf{0} \leq \mathbf{g}_{\mathbf{q},n+1}\mathbf{v}_{n+1} + e\,\mathbf{g}_{\mathbf{q},n}\mathbf{v}_n \perp \boldsymbol{\Lambda}_{n+1} \geq 0 \tag{14}$$

which can be written in an equation as

$$\text{if } \mathbf{g}_n \leq \mathbf{0} \text{ then } \boldsymbol{\Lambda}_{n+1} - \max(\mathbf{0}, \boldsymbol{\Lambda}_{n+1} - (\mathbf{g}_{\mathbf{q},n+1}\mathbf{v}_{n+1} + e\,\mathbf{g}_{\mathbf{q},n}\mathbf{v}_n)) = \mathbf{0} \tag{15}$$

Question by Olivier to Vincent: in the above two equations is it "if $\mathbf{g}_n \leq \mathbf{0}$" or "if $\mathbf{g}_{n+1} \leq \mathbf{0}$"? Therefore the discretized form of the equation of motions are

$$\begin{array}{rcl}\mathbf{M}\mathbf{a}_{n+1} & = & \mathbf{h}_{n+1} \\ \mathbf{M}\Delta\mathbf{v}_{n+1}^{\mathrm{con}} & = & \mathbf{g}_{\mathbf{q}}^T\boldsymbol{\Lambda}_{n+1} \\ \mathbf{0} & = & \boldsymbol{\Phi}_{n+1} \\ \mathbf{0} & = & \boldsymbol{\Lambda}_{n+1} - \max(\mathbf{0}, \boldsymbol{\Lambda}_{n+1} - (\mathbf{g}_{\mathbf{q},n+1}\mathbf{v}_{n+1} + e\,\mathbf{g}_{\mathbf{q},n}\mathbf{v}_n))\end{array} \tag{16}$$

Question by Olivier to Vincent: shouldn't we restrict the last equations to the *active* set of unilateral constraints?

For smooth systems, the classical HHT method can be obtained by introducing a vector $\mathbf{A}$ of acceleration-like variables and by using Newmark formulae as:

$$\mathbf{q}_{n+1} = \mathbf{q}_n + h\mathbf{v}_n + h^2(0.5-\beta)\mathbf{A}_n + h^2\beta\mathbf{A}_{n+1} \tag{17}$$

$$\mathbf{v}_{n+1} = \mathbf{v}_n + h(1-\gamma)\mathbf{A}_n + h\gamma\mathbf{A}_{n+1} \tag{18}$$

$$\mathbf{A}_{n+1} = (1-\alpha_f)\mathbf{a}_{n+1} + \alpha_f\mathbf{a}_n \tag{19}$$

with $\gamma = 0.5 + \alpha_f$, $\beta = 0.25(\gamma + 0.5)^2$, $\alpha_f \in [0, 1/3]$. Furthermore, the variables $\mathbf{A}_n$ and $\mathbf{A}_{n+1}$ can be eliminated in Eqs. (17-19), so that a two-step recurrence is obtained:

$$\begin{aligned}
\mathbf{v}_{n+1} &= \mathbf{v}_n + \alpha_f(1-\gamma)h\mathbf{a}_{n-1} + ((1-\alpha_f)(1-\gamma) + \alpha_f\gamma)\, h\mathbf{a}_n + (1-\alpha_f)\gamma h\mathbf{a}_{n+1} && (20)\\
\mathbf{q}_{n+1} &= \mathbf{q}_n + h\mathbf{v}_n + \alpha_f(0.5-\beta)h^2\mathbf{a}_{n-1} + ((1-\alpha_f)(0.5-\beta) + \alpha_f\beta)\, h^2\mathbf{a}_n \\
&\quad +(1-\alpha_f)\beta h^2\mathbf{a}_{n+1} && (21)
\end{aligned}$$

If non-smooth impacts occur, a possible extension of this algorithm could be

$$\begin{aligned}
\mathbf{v}_{n+1} &= \mathbf{v}_n + \alpha_f(1-\gamma)(h\mathbf{a}_{n-1} + \Delta\mathbf{v}^{\text{con}}_{n-1}) + ((1-\alpha_f)(1-\gamma) + \alpha_f\gamma)\,(h\mathbf{a}_n + \Delta\mathbf{v}^{\text{con}}_n) \\
&\quad +(1-\alpha_f)\gamma(h\mathbf{a}_{n+1} + \Delta\mathbf{v}^{\text{con}}_{n+1}) && (22)\\
\mathbf{q}_{n+1} &= \mathbf{q}_n + h\mathbf{v}_n + \alpha_f(0.5-\beta)h(h\mathbf{a}_{n-1} + \Delta\mathbf{v}^{\text{con}}_{n-1}) \\
&\quad + ((1-\alpha_f)(0.5-\beta) + \alpha_f\beta)\, h(h\mathbf{a}_n + \Delta\mathbf{v}^{\text{con}}_n) + (1-\alpha_f)\beta h(h\mathbf{a}_{n+1} + \Delta\mathbf{v}^{\text{con}}_{n+1}) && (23)
\end{aligned}$$

These formulae result from the direct application of the HHT scheme to the equations of motion with unilateral constraints. However, as it will be demonstrated in the examples below, this algorithm turns out to be consistent only in if $\gamma = 1$ and $\alpha_f = 0$, but it lacks consistency otherwise. This method is only presented for comparison with the nonsmooth HHT method which is detailed in the following and which allows a consistent treatment of impacts during motion. For the sake of comparison, numerical experiments will also be done using the classical HHT algorithm with a formulation of the unilateral constraints at position level, i.e. the last equation in Eq. (16) is replaced by the condition $\mathbf{0} = \mathbf{\Lambda}_{n+1} - \max(\mathbf{0}, \mathbf{\Lambda}_{n+1} - \mathbf{g}(\mathbf{q}_{n+1}))$. It will be shown that this strategy also lacks consistency.

The consistent nonsmooth HHT method is now explained. Suppose that an impact is detected at time $t_i \in [t_n, t_{n+1}]$, the impact reaction then occurs only in the atomic measure at the instant t_i. The influence of the impact is then considered only at the current time step and not at any other time steps. The contribution of the nonsmooth force to the velocity and the position is then derived as:

$$\begin{aligned}
\Delta\mathbf{v}^{con}_{n+1} &= \int_{[t_n, t_{n+1}]} \mathrm{d}\mathbf{v}^{con} && (24)\\
\Delta\mathbf{q}^{con}_{n+1} &= \int_{[t_n, t_{n+1}]}\int_{[t_n, t_{n+1}]} \mathrm{d}\mathbf{v}^{con}\,\mathrm{d}t = 0.5\, h\, \Delta\mathbf{v}^{con}_{n+1} && (25)
\end{aligned}$$

where $\Delta\mathbf{v}^{con}_{n+1}$ also satisfies the discrete equations of motion (16). The nonsmooth HHT method is inspired by a combination of these formulae with the classical HHT formulae and is formulated as

$$\begin{aligned}
\mathbf{v}_{n+1} &= \mathbf{v}_n + \alpha_f(1-\gamma)h\mathbf{a}_{n-1} + ((1-\alpha_f)(1-\gamma) + \alpha_f\gamma)\, h\mathbf{a}_n + (1-\alpha_f)\gamma h\mathbf{a}_{n+1} \\
&\quad +\Delta\mathbf{v}^{con}_{n+1} && (26)\\
\mathbf{q}_{n+1} &= \mathbf{q}_n + h\mathbf{v}_n + \alpha_f(0.5-\beta)h^2\mathbf{a}_{n-1} + ((1-\alpha_f)(0.5-\beta) + \alpha_f\beta)\, h^2\mathbf{a}_n \\
&\quad +(1-\alpha_f)\beta h^2\mathbf{a}_{n+1} + 0.5\, h\, \Delta\mathbf{v}^{con}_{n+1} && (27)
\end{aligned}$$

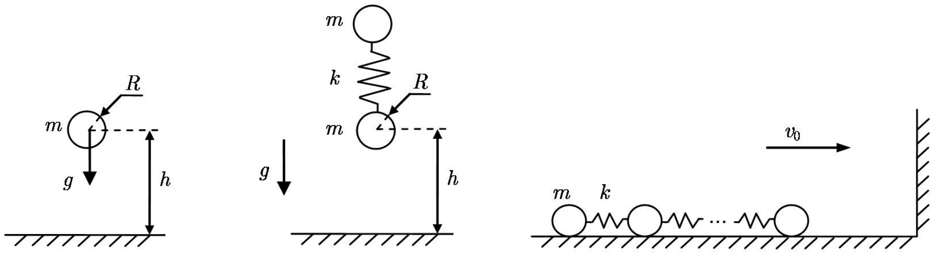
These formulae can be solved together with the discrete equations of motion using a predictor-corrector scheme coupled with a solver for linear complementarity problems. The linearized equation for Newton iteration becomes:

$$\begin{bmatrix}
\mathbf{M} + h(1-\alpha_f)\gamma\mathbf{C}_T + h^2(1-\alpha_f)\beta\mathbf{K}_T & \mathbf{C}_T + 0.5h\mathbf{K}_T & \mathbf{\Phi}_{\mathbf{q}}^T & \mathbf{0} \\
\mathbf{0} & \mathbf{M} & \mathbf{0} & -\mathbf{g}_{\mathbf{q}}^T \\
h^2(1-\alpha_f)\beta\mathbf{\Phi}_{\mathbf{q}} & 0.5h\mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} & \mathbf{0} \\
h(1-\alpha_f)\gamma\mathbf{g}_{\mathbf{q}} & \mathbf{g}_{\mathbf{q}} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta\mathbf{a} \\ \Delta(\Delta\mathbf{v}^{con}) \\ \Delta\boldsymbol{\lambda}_b \\ \Delta\mathbf{\Lambda}_u
\end{bmatrix}
=
\begin{bmatrix}
-\mathbf{res}^{\mathbf{q}} \\ -\mathbf{res}^{\mathbf{con}} \\ -\mathbf{res}^{\mathbf{\Phi}} \\ -\mathbf{res}^{\mathbf{g}}
\end{bmatrix} \quad (28)$$

where the tangent stiffness and damping matrices are defined as: $\mathbf{C}_T = -\dfrac{\partial h}{\partial v}$ and $\mathbf{K}_T = \dfrac{\partial}{\partial \mathbf{q}}(\mathbf{M}\mathbf{a} - \mathbf{h})$, and $\mathbf{res}^{\mathbf{q}}$, $\mathbf{res}^{\mathbf{con}}$, $\mathbf{res}^{\mathbf{\Phi}}$, $\mathbf{res}^{\mathbf{g}}$ denote the residuals of the four subsets of equations in Eq. (16).

4 Numerical examples

Examples are given in this section to test the numerical methods. In order to highlight the properties of the presented time integration method, examples for both rigid-body systems and flexible-body systems are considered, as shown in Fig. 1.

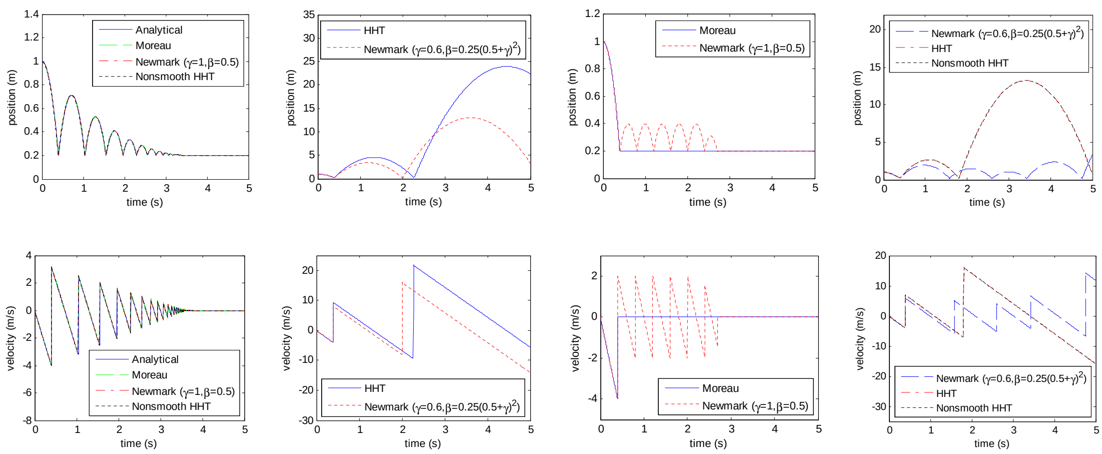


(a) (b) (c)

Figure 1. Examples: (a) bouncing ball; (b) linear vertical oscillator; (c) a chain of oscillators.

4.1 Bouncing ball

The first example is a standard bouncing ball on a smooth plane, as depicted in Fig. 1(a). The bouncing ball is a well known benchmark example in the field of nonsmooth mechanics. As usual, both the ball and the plane are rigid-body models, and the model is of one degree of freedom (DOF). The ball is subjected to the gravity, bounces on the rigid plane with a restitution coefficient of $e = 0.8$. Thus, it introduces a unilateral constraint on the vertical position of the ball. The ball is standstill in the beginning. Other physical parameters of this example are as follows: mass $m = 1kg$, radius $R = 0.2m$, gravity accelerating $g = 10m/s^2$, initial height $h = 1.001m$.



(a) (b) (c) (d)

Figure 2. Position and velocity of the bouncing ball vs. time: (a,b) numerical methods on velocity-level constraints: $\mathbf{g}_{\mathbf{q},n+1}\mathbf{v}_{n+1} + e\,\mathbf{g}_{\mathbf{q},n}\mathbf{v}_n = 0$; (c,d) comparison of numerical methods on position-level constraints: $\mathbf{g}(\mathbf{q}) = 0$.

The numerical parameters are set as: time step of $10^{-3}s$ for all the methods; for the nonsmooth HHT method and the classical HHT method, α_f is chosen as 0.1, for the Moreau-Jean time stepping method, $\theta = 1$. Other parameters are listed in Fig. 2.

As one can see from the figure, only the numerical methods with velocity constraint shown in Fig. 2(a) are valid. Newmark method is stable only if it is fully implicit, i.e., $\gamma = 1$, $\beta = 0.5$. The time integration methods on position-level constraint are either divergent or sensitive to the initial physical parameters, e.g., the fully implicit Newmark method on position-level constraint does not show consistent impact characteristics. Therefore, the methods on position constraint are indeed not reliable.

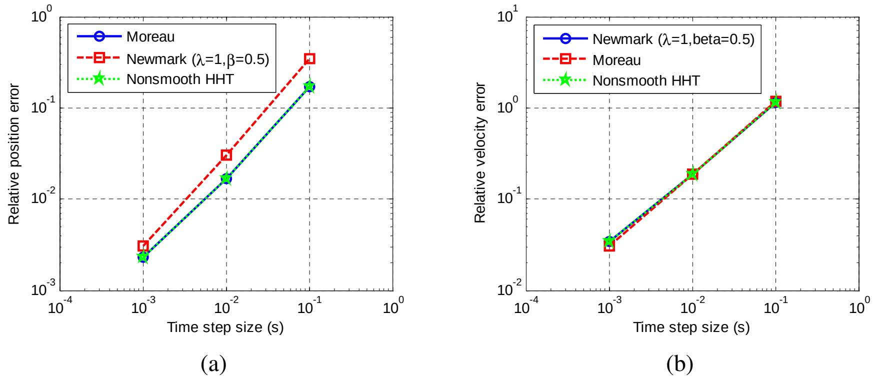


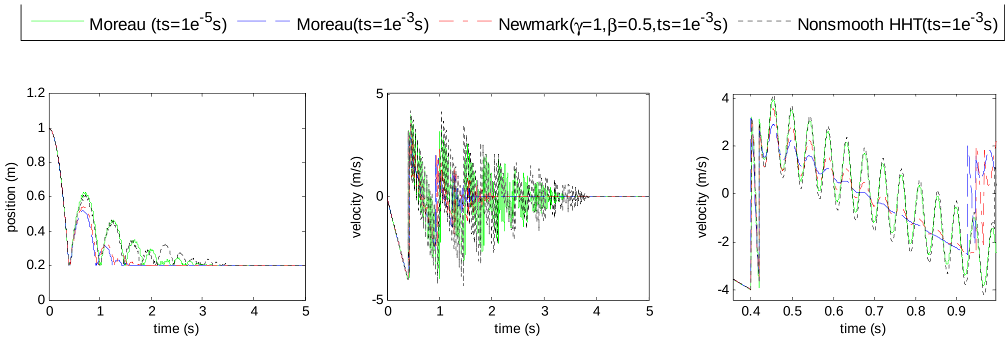
Figure 3. Comparison of relative errors on norm L^1 form: (a) position, (b) velocity.

Figure 3 shows the convergence analysis of the valid methods. Relative errors are computed on the L^1 form, see [15]. Moreau, fully implicit Newmark and nonsmooth HHT methods all remain first order in the overall range. However, Newmark and nonsmooth HHT methods have slightly better accuracy in position response.

4.2 Linear oscillator

Here in this example, a vertical linear oscillator example is studied. The model of the oscillator is shown in Fig. 1(b), where two masses are connected by a spring. The oscillator is subjected to the gravity, and also has one DOF in the vertical direction. After the lower ball bounces on the smooth plane, it bounces with a restitution coefficient of 0.8. In the meanwhile, it is also subjected by the compressed spring. Thus, a second impact or multiple impacts could happen. In the free-flight mode, the system is oscillating with its natural frequency. The mass is of 1kg for each ball, and the stiffness of the spring is $10^4 N/m$. The initial velocity is zero and the initial height of the bottom mass is of $1.001m$.

This example is simulated using the three validated methods. Firstly, the time step size is set as $t_s = 10^{-3}s$ for all the three methods, and then a finer time step $t_s = 10^{-5}s$ is used for Moreau time-stepping method for a comparison. Other numerical parameters are chosen to be the same as that in the first example. Results in Fig. 4 shows the position, velocity of the bottom mass respectively, and Figure 5 shows the total energy dissipation of the oscillator system. For all the three methods, double impacts occurs due to the compression of the spring. During the free flight after the first double impacts, the system is oscillating with slight energy dissipation using the nonsmooth HHT method; however, the energy dissipation tends to be comparatively large for the fully implicit Newmark and Moreau methods with the same time step size. Decreasing the time step size for Moreau method, one finds out that the energy dissipation is smaller and the results tend to be closer to the nonsmooth HHT method. Thus, one can conclude that nonsmooth HHT method has better quality of global accuracy compared to the other two methods.



(a) (b) (c)

Figure 4. Comparison the numerical results for the linear vertical oscillator: (a) position, (b,c) velocity.

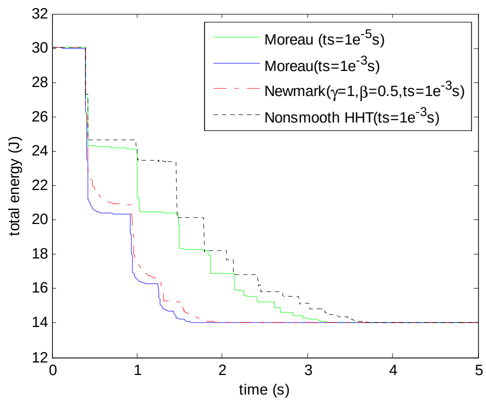


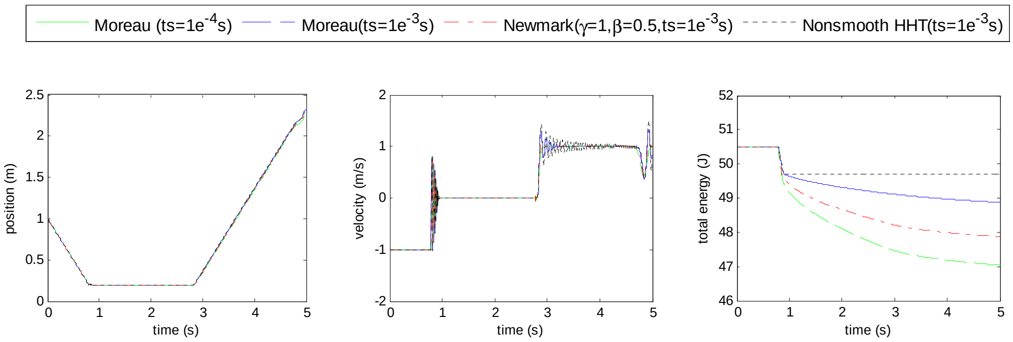
Figure 5. Numerical results for the total energy of the linear vertical oscillator.

4.3 A chain of oscillators

The example is studied in order to see the numerical results when a close contact occurs. The chain of oscillator model is placed on a smooth plane with any friction neglected, see Fig. 1(c). It moves toward a wall on the left, and bounces back after impact. This model comprises of 100 masses and 99 springs, which connect the masses one by one. The initial velocity of chain is set as $v_0 = -1m/s$, with the positive direction to the left. The rest of the physical parameters remain the same as the above two examples. Figure 6 shows the position and velocity of the mass on the right, where an impact could occur. Also, the total energy of the oscillator chain is analyzed, as shown in Fig. 6(c). As one can tell from the figures, close contact analysis are stable for all the three methods. For the smooth motion of the other parts of the oscillators, nonsmooth HHT method has least energy dissipation. Choosing a smaller time step size for Moreau method, one gets closer energy dissipation as that of the nonsmooth HHT one. The results further prove that nonsmooth HHT has better accuracy properties than the other two methods, as already shown in Section 4.2.

5 CONCLUSIONS

In the field of smooth flexible multibody dynamics, HHT and the generalized-α time integration methods have been successfully carried out with the characteristics of second-order accuracy and unconditional stability in the linear regime. Considering that in reality, many mechanical systems are subjected to both bilateral constraints and unilateral constraints, the authors aim at extending the HHT method to include the nonsmooth characteristics of unilateral constraints. The method presented here is denoted by the nonsmooth



(a) (b) (c)

Figure 6. Comparison the numerical results for the chain of oscillators: (a) position, (b) velocity, (c) total energy.

HHT method. It is of the purpose to benefit from the qualitative properties of the HHT method in terms of accuracy, stability and the controllable numerical damping.

Numerical examples are studied and results show that the Moreau, the fully implicit Newmark and the nonsmooth HHT methods are valid with unilateral constraints on velocity level. Based on the comparison through the examples of both rigid and flexible system models, the methods are stable and the presented nonsmooth HHT method has considerably better quality of accuracy, especially for the flexible system dynamics. Therefore, we conclude that the nonsmooth HHT time integration method can substantially decrease the computational cost for major models, such as structural dynamics of wind turbine systems, in which unilateral contacts might occur.

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