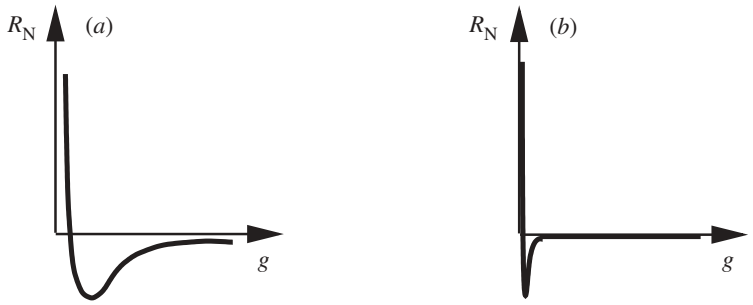

Figure 1. Lennard–Jones graph.

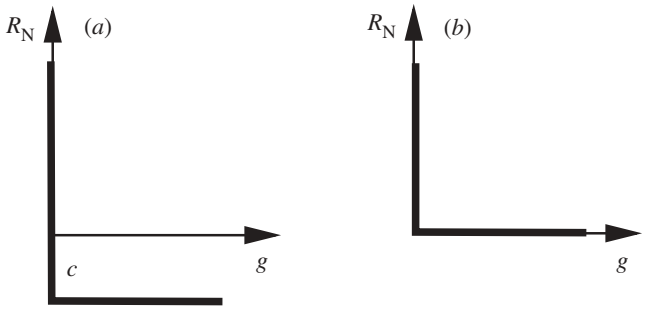

Figure 2. 'Non-smooth' cohesive law: (*a*) with cohesive status; (*b*) without cohesive status.

candidate point and the antagonist body boundary, is positive, $g \geqslant 0$. Together with the fact that the reaction force should have a positive normal component $R_N \geqslant 0$, vanishing if the gap is strictly positive, $gR_N = 0$, one arrives at the so-called Signorini condition. Another way, coming into details, is to consider that the bodies are rough, and that contact is set between elastic asperities scattered on some smooth reference boundaries surfaces. The gap g should be understood as the distance between these reference surfaces, and might take negative values when asperities are deformed. The normal component of the reaction force might appear to depend linearly on the penetration distance, $R_N = -kg$, where k is some stiffness related to the elastic properties of the asperities. The normal reaction force vanishes, if the gap is positive. The stiffness k should be large enough to forbid too large penetration, so as to satisfy some 'macroscopic' impenetrability.

In many circumstances, the adopted unilaterality description does not matter, provided that the penetration remains reasonably small. Most often, it is a matter of numerical convenience to adopt one or other law, the flexibility model, $R_N = -kg$, etc., being a regularized form of the Signorini condition, allowing us to use those special methods of non-smooth mechanics, which are methods adapted from smooth mechanics. The above discussion, about unilaterality, holds for dry friction. Coulomb's law accounts for the main features of dry friction, i.e. sliding requires that tangential efforts exceed some threshold, the larger the normal pressure, the larger the threshold. Regularized forms of Coulomb's law may be relevant as well, and more sophisticated laws might be used, specially if tribological data are available.

A cohesion phenomenon may be defined as the faculty of two bodies to remain i

contact as long as separating forces do not exceed some threshold. There exist many physical mechanisms producing such an effect. At molecular scales, the Lennard–Jones law is commonly used, accounting for impenetrability, and describing tensile forces rapidly vanishing when the gap is increased (see figure 1*a* and Rose *et al.* (1981, 1983)). The relation between the normal reaction force and the gap is smooth but steep. Following the line of the above discussion, one may ignore those details describing the cohesive forces near the threshold at a fine scale and adopt a graph such as figure 1*b*, which may be viewed as an image of figure 1*a*, resulting from a contraction of the gap scale. One might keep in mind the fact that the gap is positive, that the normal reaction force may not exceed some negative threshold, and that this force vanishes when the gap is strictly positive, data to be summarized on a 'limit graph'. In fact, a limit graph will not contain full data, and energy estimation is ambiguous. One is led to distinguish the history of the contacting bodies or *left status* before the contacting instant, and the *right status*,

if the left status is 'cohesive':
$g \geqslant 0, \quad R_{\mathrm{N}} + c \geqslant 0, \quad g(R_{\mathrm{N}} + c) = 0,$
else:
$g \geqslant 0, \quad R_{\mathrm{N}} \geqslant 0, \quad gR_{\mathrm{N}} = 0,$

the graphs of these relations being depicted in figure 2. These relations are written together with status change rules:

the right status is the same as the left status,
except
(a) if the left status is 'cohesive',
and if the solution R_{N} satisfies $R_{\mathrm{N}} + c < 0$ or $g > 0$,
then the right status is set to 'non-cohesive';
(b) if the left status is 'non-cohesive',
and if the solution satisfies $g \leqslant 0$,
then the right status is set to 'cohesive'.

This rule allows the contacting bodies to be glued again. Other status definitions and change rules may be introduced. For instance, the following rule allows bodies to be glued again only when some pressure is exerted:

(b) if the left status is 'non-cohesive',
and if the solution satisfies $g \leqslant 0$ and $R_{\mathrm{N}} > 0$,
then the right status is set to 'cohesive'.

Rule (b) may be omitted, which means that when separated, bodies cannot be glued again. More sophisticated cohesive models will be referred to in § 2.

In the preceding, steep regular models were presented against strictly non-smooth (non-differentiable) models, for instance, the flexibility model of unilateral constraints against the Signorini relation, the Lennard–Jones model against the Signorini-like condition monitored by the cohesive status. Though regularized models allow us to use the facilities of smooth methods, adapted for the circumstances, it happens

Clarify sentence?

that mere changes of variables allow us to apply a 'standard non-smooth contact dynamics (NSCD) method' to a wide range of models, so-called 'Signorini, Coulomb, derived' models, either steep regular models or strictly non-smooth models, as will be developed in § 6.

The NSCD method (Jean 1995, 1999; Jean & Moreau 1992; Moreau 1994, 1999) does not use the facilities of regularized formulations, and rather in the line of convex analysis, uses a pair of relations: the Signorini condition (a complementary relation) and Coulomb's law. With this respect, the NSCD method has some relationship with Newton's generalized method (Alart & Curnier 1988, 1991), mathematical programming methods (Chabrand *et al.* 1995; Klarbring 1990), and the bi-potential method (De Saxcé & Feng 1991). Nevertheless, the NSCD solving procedure differs from those used in these methods. The main features of the standard NSCD method, are

(i) for each candidate to contact (a unique pair composed of a candidate body and of an antagonist body), the relative velocity and the reaction force are related through a unilateral Signorini-like relation and a frictional Coulomb-like relation;

(ii) the relative velocity and the reaction force are also linearily related through a linearized form of the dynamical equation;

(iii) for each candidate to contact, assuming provisional values of the reaction forces at other candidates, a straightforward solution may be computed for the so-called 'Signorini, Coulomb, standard' problem defined in the above items;

(iv) the unknowns are updated from a candidate to the next candidate using the 'standard' solution; this updating procedure is similar to a nonlinear block Gauss–Seidel algorithm.

More details will be found in § 5 *a*. It is noteworthy that the NSCD method is fully implicit, while the usual regularized methods are fully explicit.

2. Different approaches of cohesive laws

Some particular cohesive laws are the so-called 'cohesive zone models'. These models relating reaction forces and displacement jumps across an interface, between two candidate bodies, or two candidate parts of the same body, are commonly used to describe the separation of some glued materials as well as the initiation and propagation of cracks. Fracture mechanics and cohesive zone models are two main approaches for analysing these crack initiations and propagations in materials.

Fracture mechanics is a classical tool of investigation, and particularly successful when applied to homogeneous elastic materials in quasi-static situations. It might fail in more complex circumstances, mainly when the question of initiation of cracks, space (branching) instability and time instability while growing arises. It might suffer also from the lack of ability to take into account complex physical phenomena such as frictional contact between the edges of a crack, wear, growth, coalescence, etc. Much effort has been put into these areas of research. But so far, either theoretically or numerically, describing a complete process of birth and growth of cracks is still an open problem. Nevertheless, fracture mechanics offers relevant energetic considerations, in particular, the energy release rate G.

The cohesive zone models allow us to attack the question of birth and growth of cracks in complex situations, composite materials, highly heterogeneous microstructured materials, complex local loadings, three-dimensional cracks, instabilities, etc., while keeping in mind the physics at the local scale. Cohesive zone models allow us to mix several kinds of energy criterion, such as the critical energy release rate and the maximal stress criterion (mode II maximal shearing, etc.). Crack growth is described with those kinds of local criteria written with stresses and displacement jumps. Initiation and propagation should result from the knowledge of the loading path, without supplementary assumptions.

Cohesive zone models, initially introduced by Dugdale (1960), Barenblatt (1962) and Rice (1968), involved only the relations between normal stresses (traction or compression) and the opening (gap) at the edges of a crack. Later on, shearing was taken into account in the models of Ida (1972) on dynamical fracture in the Earth's crust and of Palmer & Rice (1973) on the shearing of an inclined clay layer under gravity. Unilateral conditions were introduced later by Frémond (1982). The concept of cohesive domains was generalized in the late 1980s by Needleman (1987), who proposed phenomenological models of decohesion, independent of the loading path, suggested by the atomic scale researches of Rose *et al.* (1981, 1983) on bimetallic materials. These models, ready to become realistic crack models, take into account, as soon as 1990, the local irreversible behaviour, introducing surface damage (Needleman 1992; Tvergaard 1990). Considering the kinematics of the opening, in the normal direction *as well as* in the tangential direction (Tvergaard 1990) introduces a post-decohesion Coulomb frictional behaviour, and the bounding effect of residual stresses (Tvergaard 1991). After these pioneering works, recent analytical and numerical developments deal with quasi-static or dynamic propagation of cracks (Costanzo & Walton 1997; Needleman & Rosakis 1999; Xu & Needleman 1994; Xu *et al.* 1998). A pair composed of an antagonist point and a candidate point to contact, in a discrete formulation, might be considered as well as a cohesive zone model. The cohesive zone model not only applies to cracks but also to bulk material. This idea has been applied in quite general situations: elasto-plasticity (Needleman 1990; Tvergaard & Hutchinson 1992), dynamical propagation of cracks (Xu & Needleman 1994), elasto-visco-plasticity (Siegmund & Needleman 1997), and composite materials (Needleman & Rosakis 1999; Siegmund *et al.* 1997; Xu *et al.* 1997). One may also quote the contributions of the French school on the micromechanical approach of cracks in composite materials (Cangémi *et al.* 1996; Chaboche *et al.* 1997; Frémond 1982; Michel & Suquet 1994), and on damage in composite material plies (Allix *et al.* 1989, 1995).

3. The basic equations

Though emphasis is put on deformable bodies, the question of space discretization is not discussed in this paper. Finite-elements methods are used. Discrete variables and equations governing the coordinates of the nodes will be written at once.

(*a*) *Notation*

For the sake of simplicity, it will be assumed that some nodes on a meshed body candidate to contact are marked as candidate material points. To each candidate P

is associated a unit vector $\boldsymbol{n}$ orthogonal to the boundary of the antagonist body, directed from the antagonist body to the candidate body. The material point P′ lying on the boundary of the antagonist body, and such that $\overrightarrow{\mathrm{P'P}}$ is collinear to $\boldsymbol{n}$ (i.e. the orthogonal projection of P on the antagonist boundary), is marked as the antagonist material point associated to P. An approximate boundary, or an approximate normal vector is often used, for convenience, and to ensure the uniqueness of the antagonist material point. In tricky circumstances, any reasonable algorithm to define an antagonist material point P′, 'proximal' to P, will do. It will be referred to as *a candidate to contact*, as the unique pair of candidate antagonist bodies, or as the candidate antagonist material point, or merely as the candidate material point. An orthonormal basis $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{s})$, referred to as the local frame, may be constructed with any two unit vectors $\boldsymbol{t}$, $\boldsymbol{s}$, orthogonal to each other and to $\boldsymbol{n}$. Normal components of vectors in the local frame will be denoted by the subscript 'N', while the pair of tangential components will be denoted by the subscript 'T'. The relative velocity vector (in an extended sense) is defined as $\boldsymbol{U} = \boldsymbol{V}(\mathrm{P}) - \boldsymbol{V}(\mathrm{P}')$, where $\boldsymbol{V}(\mathrm{P})$ is the velocity vector of the material point P, and $\boldsymbol{V}(\mathrm{P}')$ is the velocity vector of the material point P′. The components of this vector in the local frame will be written as $U = (U_{\mathrm{T}}, U_{\mathrm{N}})$. The reaction force (dual variable of $\boldsymbol{U}$), acting from the antagonist body to the candidate body, has components $R = (R_{\mathrm{T}}, R_{\mathrm{N}})$ in the local frame. The gap is defined as $g = \overline{\mathrm{P'P}}$. It is strictly positive when bodies are not contacting, and strictly negative when they interpenetrate. The candidates to contact will be labelled with Greek superscripts, such as α and β. The number of candidates to contact is χ.

'for'?

(*b*) *Kinematic relations*

The variable $q \in \mathbb{R}^N$ is the (column) vector of nodes coordinates, n being the number of nodes, $N = 3n$ being the number of degrees of freedom. If some degrees of freedom are imposed, penalization techniques are used. During motion, q is a function of the time t. The time derivative of the mapping $t \to q(t)$ is denoted $\dot{q}$. There exists a linear mapping $G^\alpha(q)$ such that the relative velocity at the candidate α is given by the formula,

$$U^\alpha = G^\alpha(q)\dot{q}.$$

For instance, if P is a node candidate to contact, P′ the antagonist particle lying on a mesh boundary line with end points the nodes A, B, the velocity of P′ is $\boldsymbol{V}(\mathrm{P}') = (\overline{\mathrm{AP}'}/\overline{\mathrm{AB}})\boldsymbol{V}(\mathrm{B}) + (\overline{\mathrm{P'B}}/\overline{\mathrm{AB}})\boldsymbol{V}(\mathrm{A})$, and $\boldsymbol{U} = \boldsymbol{V}(\mathrm{P}) - \boldsymbol{V}(\mathrm{P}')$, which yields a formula as above. From duality considerations (conservation of the power), the node resulting force r^α corresponding to a local reaction force R^α satisfies

$$\forall v, \quad G^\alpha(q)v \cdot R^\alpha = v \cdot G^{*\alpha}(q)R^\alpha = v \cdot r^\alpha,$$

where $G^{*\alpha}(q) = H^\alpha(q)$ is the linear transposed mapping of $G^\alpha(q)$. The usual notation in this paper is

$$U^\alpha = H^{*\alpha}(q)\dot{q}, \qquad r^\alpha = H^\alpha(q)R^\alpha. \tag{3.1}$$

Another fundamental kinematic formula is that the derivative of the gap function $t \to g(t)$ is the normal component of the relative velocity,

$$\dot{g} = U_{\mathrm{N}}. \tag{3.2}$$

(*c*) *The dynamical equation*

The dynamical equation of the problem is written,

$$M\ddot{q} = F_{\text{int}}(q, \dot{q}) + P(t) + r, \tag{3.3}$$

where M is the mass matrix, $F_{\text{int}}(q, \dot{q})$ represents the node forces corresponding to internal efforts, $P(t)$ represents explicitly known excitation forces, and r are the node forces corresponding to the reaction forces exerted at candidates to contact.

(*d*) *The Signorini relation and the frictional Coulomb's law*

The unilateral Signorini condition is written as

$$g \geqslant 0, \quad R_{\text{N}} \geqslant 0, \quad gR_{\text{N}} = 0. \tag{3.4}$$

Using 'standard variables', $\mathcal{U}_{\text{N}}$, $\mathcal{R}_{\text{N}}$, the complementary relation,

$$\mathcal{U}_{\text{N}} \geqslant 0, \quad \mathcal{R}_{\text{N}} \geqslant 0, \quad \mathcal{U}_{\text{N}}\mathcal{R}_{\text{N}} = 0,$$

or any equivalent form, will be referred to as the 'Signorini standard' relation,

$$S_{\text{Sig}}(\mathcal{U}_{\text{N}}, \mathcal{R}_{\text{N}}). \tag{3.5}$$

An equivalent form is

$$\mathcal{R}_{\text{N}} = \text{proj}_{\mathbb{R}^+}(\mathcal{R}_{\text{N}} - \rho\mathcal{U}_{\text{N}}), \quad \text{where } \rho > 0 \text{ is arbitrary.} \tag{3.6}$$

Coulomb's law is written as

$$\|R_{\text{T}}\| \leqslant \mu R_{\text{N}}, \quad \|U_{\text{T}}\| \neq 0 \Rightarrow R_{\text{T}} = -\mu R_{\text{N}} \frac{U_{\text{T}}}{\|U_{\text{T}}\|}. \tag{3.7}$$

Using 'standard variables' $\mathcal{U}$, $\mathcal{R}$, the relation,

$$\|\mathcal{R}_{\text{T}}\| \leqslant \mu\mathcal{R}_{\text{N}}, \quad \|\mathcal{U}_{\text{T}}\| \neq 0 \Rightarrow \mathcal{R}_{\text{T}} = -\mu\mathcal{R}_{\text{N}} \frac{\mathcal{U}_{\text{T}}}{\|\mathcal{U}_{\text{T}}\|},$$

or any equivalent form, will be referred to as the 'μ-Coulomb standard' law,

$$C_{\text{Coul}\,\mu\mathcal{R}_{\text{N}}}(\mathcal{U}_{\text{T}}, \mathcal{R}_{\text{T}}). \tag{3.8}$$

An equivalent form is

$$\mathcal{R}_{\text{T}} = \text{proj}_{\mu\mathcal{R}_{\text{N}}\mathbb{B}}(\mathcal{R}_{\text{T}} - \rho\mathcal{U}_{\text{T}}), \quad \text{where } \rho > 0 \text{ is arbitrary.} \tag{3.9}$$

$\mathbb{B}$ is the unit disc with centre 0 in $\mathbb{R}^2$. Another equivalent form is expressed as a principle of minimal dissipation,

$$\mathcal{R}_{\text{T}} \in \mu\mathcal{R}_{\text{N}}\mathbb{B}, \quad \forall\mathcal{S} \in \mu\mathcal{R}_{\text{N}}\mathbb{B}, \quad \mathcal{U}_{\text{T}} \cdot (\mathcal{S} - \mathcal{R}_{\text{T}}) \geqslant 0. \tag{3.10}$$

4. Time discretization

(a) Numerical algorithm for the dynamical equation

When time discretization is performed, an elementary subinterval $]t_i, t_{i+1}]$ of length h is considered. The main idea developed in the process of time discretization is that discrete variables are not necessarily to be defined at some special time belonging to this interval, since the times where significant frictional effects occur are usually unknown, or costly to approximate, or even difficult to separate, when simultaneous contacts are occurring. Discrete frictional contact relations are thus defined 'over' the interval of time $]t_i, t_{i+1}]$. Integrating both sides of the dynamical equation yields

$$\left.\begin{aligned} M(\dot{q}(t_{i+1}) - \dot{q}(t_i)) &= \int_{t_i}^{t_{i+1}} F(q,\dot{q})\,\mathrm{d}s + \int_{]t_i,t_{i+1}]} P\,\mathrm{d}t + \int_{]t_i,t_{i+1}]} r\,\mathrm{d}\nu, \\ q(t_{i+1}) &= q(t_i) + \int_{t_i}^{t_{i+1}} \dot{q}\,\mathrm{d}s. \end{aligned}\right\} \tag{4.1}$$

The *mean value impulse* denoted $r(i+1)$,

$$r(i+1) = \frac{1}{h}\int_{]t_i,t_{i+1}]} r\,\mathrm{d}\nu, \tag{4.2}$$

emerges as a primary unknown. A typical numerical method is the θ-method. Setting $\dot{q}(i)$, $q(i)$, $\dot{q}(i+1)$, $q(i+1)$, approximations of $\dot{q}(t_i)$, $q(t_i)$, $\dot{q}(t_{i+1})$, $q(t_{i+1})$, are OK?

$$\int_{t_i}^{t_i+1} F(q,\dot{q})\,\mathrm{d}s \approx h\theta F(q(i+1),\dot{q}(i+1)) + h(1-\theta)F(q(i),\dot{q}(i)),$$
$$q(i+1) = q(i) + h\theta\dot{q}(i+1) + h(1-\theta)\dot{q}(i).$$

Since, in this paper, emphasis is put on cohesive frictional laws, it will be supposed that materials are elastic within the assumption of small perturbations. Thus the technical details of nesting some Newton–Raphson loop will be avoided, and introduced only if the material behaviour law is nonlinear (see Jean 1999). Assuming an elastic linear behaviour,

$$F(q,\dot{q}) = -V\dot{q} - Kq,$$

where V is the damping matrix and K is the stiffness matrix, equation (4.1) is written as

$$\begin{aligned} M(\dot{q}(i+1) - \dot{q}(i)) = h\theta(&-V\dot{q}(i+1) - Kq(i+1) + P(t_{i+1})) \\ &+ h(1-\theta)(-V\dot{q}(i) - Kq(i) + P(t_i)) + hr(i+1), \end{aligned}$$

and the θ-algorithm is

$$\left.\begin{aligned} \dot{q}(i+1) - \dot{q}(i) &= w(-hV\dot{q}(i) - hK(q(i) + \theta h\dot{q}(i)) + hP(i+1) + hr(i+1)), \\ q(i+1) &= q(i) + h\theta\dot{q}(i+1) + h(1-\theta)\dot{q}(i), \\ &\text{where} \\ w &= (M + h\theta V + h^2\theta^2 K)^{-1}, \\ P(i+1) &= \theta P(t_{i+1}) + (1-\theta)P(t_i). \end{aligned}\right\} \tag{4.3}$$

It is assumed that the matrix $M+h\theta V+h^2\theta^2 K$ is one-to-one, which is satisfied if h is small enough, the mass matrix M being positive definite. When non-smooth mechanics effects, shocks or changes in frictional contact status are occurring, generating velocities discontinuities, first-order schemes are sufficient. Higher-order approximations schemes are irrelevant and even troublesome, since they require appropriate regularity properties which are not met. The above formula, a linearized form of the discrete dynamical equation, appears as an affine relation between two primary unknowns, the velocity $\dot{q}(i+1)$ and the mean value impulse $r(i+1)$,

$$\dot{q}(i+1) = v_{\text{free}}(i) + whr(i+1). \tag{4.4}$$

Many numerical algorithms may be written under such a form. The NSCD method is based on the use of such an affine relation, whatever the choice of the approximation process for the dynamical equation. In the following, for the sake of simplicity, the choice $\theta = 1$ is made, i.e. the choice corresponding to the implicit Euler method.

(*b*) *Discrete forms of kinematic relations*

The adopted discrete forms of (3.1) are

$$U^\alpha(i+1) = H^{*\alpha}(\hat{q})\dot{q}(i+1), \qquad r^\alpha(i+1) = H^\alpha(\hat{q})R^\alpha(i+1).$$

The variable $\hat{q}$ denotes the configuration used to define the local frames. In a fully implicit method, the choice $\hat{q} = q(i+1)$ should be made. This complication is unnecessary as far as the lengths $hU^\alpha(i+1)$ remain sufficiently small with respect to curvature radius of contacting bodies. Actually, the choice $\hat{q} = q(i)$ is easy and fair enough. In the following the variable $\hat{q}$ is omitted:

$$U^\alpha(i+1) = H^{*\alpha}\dot{q}(i+1), \qquad r^\alpha(i+1) = H^\alpha R^\alpha(i+1). \tag{4.5}$$

Denoting $\dot{g}^\alpha(i+1)$ an approximation of $\dot{g}^\alpha(t_{i+1})$, the formula (3.2) can be written,

$$\dot{g}^\alpha(i+1) = U_{\text{N}}^\alpha(i+1). \tag{4.6}$$

This formula suggests the predictive formula,

$$g^\alpha(i+1) = g^\alpha(i) + hU_{\text{N}}^\alpha(i+1). \tag{4.7}$$

The choice $\theta = 1$ has been made for the sake of simplicity. Incidentally, it is worth drawing attention to the general case $0.5 \leqslant \theta < 1$. Expressing unilateral conditions with predictive gaps at times $t_i + (1-\theta)h, t_{i+1} + (1-\theta)h$, i.e. θ-midpoints, instead of t_{i+1}, the end of the time-step, is recommended, with a predictive formula such as

$$\begin{aligned} g^\alpha(1-\theta+i) &= g^\alpha(i) + (1-\theta)hU_{\text{N}}^\alpha(i), \\ g^\alpha(1-\theta+i+1) &= g^\alpha(i+1) + (1-\theta)hU_{\text{N}}^\alpha(i+1). \end{aligned}$$

Indeed, unilateral conditions imply a contradictory rule to be satisfied both by the relative velocity and the gap at the end of the time-step (except in the case $\theta = 1$). Oscillatory artefacts might be generated. For more details see a discussion about consistency in Jean (1999) and Vola *et al.* (1998).

(c) *Discrete forms of the Signorini relation, Coulomb's law*

Clarify heading?

The Signorini relation graph is infinitely steep, and as a consequence the gap g cannot be written as a single-valued mapping of R_{N}, nor R_{N} as a single-valued mapping of g. This leads us to adopt fully implicit schemes and to choose both the values of $g^{\alpha}(i+1)$ at the end of the time-step, and $R_{\mathrm{N}}^{\alpha}(i+1)$ the mean value impulse, as unknowns in the frictional contact laws. For instance the discrete form of the Signorini relation (3.4) is

$$S_{\mathrm{Sig}}(g^{\alpha}(i+1), R_{\mathrm{N}}^{\alpha}(i+1)). \tag{4.8}$$

For similar reasons, the discrete form of Coulomb's law (3.7) is

$$C_{\mathrm{Coul}\ \mu^{\alpha} R_{\mathrm{N}}^{\alpha}(i+1)}(U_{\mathrm{T}}^{\alpha}(i+1), R_{\mathrm{T}}^{\alpha}(i+1)). \tag{4.9}$$

5. The NSCD approach

(a) *The 'Signorini, μ-Coulomb, standard' problem*

As an example consider the 'ordinary' dynamical frictional contact problem, governed by the dynamical equation (4.4), the kinematic relations (4.5), the Signorini, Coulomb's relation, (4.8), (4.9).

Using the kinematic relations allows us to write a linearized form of the dynamical equation (4.4) in terms of the local variables, the relative velocity $U^{\alpha}(i+1)$ and the reaction force $R^{\alpha}(i+1)$, at the candidate to contact α,

$$U^{\alpha}(i+1) = U_{\mathrm{free}}^{\alpha} + \sum_{\beta} W^{\alpha\beta} h R^{\beta}(i+1), \tag{5.1}$$

$$W^{\alpha\beta} = H^{*\alpha} w H^{\beta}, \tag{5.2}$$

$$U_{\mathrm{free}}^{\alpha} = H^{*\alpha} v_{\mathrm{free}}. \tag{5.3}$$

Assuming some provisional values for $R^{\beta}(i+1)$, the above equation becomes

$$U^{\alpha}(i+1) = U_{\mathrm{loc\ free}}^{\alpha} + W^{\alpha\alpha} h R^{\alpha}(i+1), \tag{5.4}$$

where

$$U_{\mathrm{loc\ free}}^{\alpha} = U_{\mathrm{free}}^{\alpha} + \sum_{\beta \neq \alpha} W^{\alpha\beta} h R^{\beta}(i+1).$$

This equation has to be written together with a discrete form of the Signorini relation, and a discrete form of Coulomb's law, together with the kinematic relation (4.7). Setting

$$\bar{U}_{\mathrm{loc\ free\ T}}^{\alpha} = U_{\mathrm{loc\ free\ T}}^{\alpha}, \qquad \bar{U}_{\mathrm{loc\ free\ N}}^{\alpha} = U_{\mathrm{loc\ free\ N}}^{\alpha} + \frac{1}{h} g^{\alpha}(i),$$
$$\bar{W}^{\alpha\alpha} = W^{\alpha\alpha}, \qquad \mu = \mu^{\alpha},$$

it is found that the unknowns,

$$\bar{U}_{\mathrm{T}} = U_{\mathrm{T}}^{\alpha}(i+1), \qquad \bar{U}_{\mathrm{N}} = \frac{1}{h} g^{\alpha}(i+1),$$
$$\bar{R} = R^{\alpha}(i+1),$$

are solutions of the following *'Signorini, μ-Coulomb, standard' problem*:

$$\left.\begin{array}{c}\text{find } \mathcal{U}, \mathcal{R}, \text{ such that} \\ \mathcal{U} = \mathcal{U}_{\text{loc free}} + \mathcal{W} h \mathcal{R}, \\ S_{\text{Sig}}(\mathcal{U}_{\text{N}}, \mathcal{R}_{\text{N}}), \\ C_{\text{Coul}\ \mu\mathcal{R}_{\text{N}}}(\mathcal{U}_{\text{T}}, \mathcal{R}_{\text{T}}),\end{array}\right\} \tag{5.5}$$

where $\mathcal{U}$, $\mathcal{R}$, $\mathcal{U}_{\text{loc free}} = \bar{U}^{\alpha}_{\text{loc free}}$ (an element of data), belong to $\mathbb{R}^3$ ($\mathbb{R}^2$ in the two-dimensional case), $\mathcal{W} = \bar{W}^{\alpha\alpha}$ (an element of data), is a 3×3 matrix (2×2 in the two-dimensional case). The solution of this elementary problem is found by discussing the intersection of piecewise affine mappings, since the Signorini relation and Coulomb's law appear in such a form (see (3.6), (3.9)). It happens that in the two-dimensional case, the elementary standard solution can be explicitly written (see § 9 and Jean (1999)). In the three-dimensional case, a few iterations of a generalized Newton–Raphson method allow super convergence toward the elementary standard solution (Alart & Curnier 1988). The NSCD algorithm is as follows.

OK?

OK?

(*b*) *Solving the frictional contact problem*

The unknowns $R^{\alpha}(i+1)$, the mean value impulses at step i, $i+1$, are sought as the limit of sequences $R^{\alpha}(p)$, $p = 1, \ldots, p_{\max}$, where $p_{\max}$ is maximum number of iterations:

(1) at iteration $p+1$, seek an approximate solution at candidate α; provisional values at other candidates are adopted as follows: if $\beta > \alpha$, these are the values computed at iteration p, and if $\beta < \alpha$, these are the values just computed at iteration $p+1$,

$$R^{\beta}(p+1),\ \beta = 1, \ldots, \alpha-1, \quad R^{\beta}(p),\ \beta = \alpha+1, \ldots, \chi;$$

the free velocity (i.e. the relative velocity at candidate α equipped with a vanishing reaction), while provisional reactions are exerted on neighbouring candidates, is computed,

$$U^{\alpha}_{\text{loc free}} = U^{\alpha}_{\text{free}} + \sum_{\beta<\alpha} W^{\alpha\beta} h R^{\beta}(p+1) + \sum_{\beta>\alpha} W^{\alpha\beta} h R^{\beta}(p),$$

(a) compute or call,

$$\bar{U}^{\alpha}_{\text{loc free T}} = U^{\alpha}_{\text{loc free T}}, \quad \bar{U}^{\alpha}_{\text{loc free N}} = U^{\alpha}_{\text{loc free N}} + \frac{1}{h} g^{\alpha}(i), \quad \bar{W} = W^{\alpha\alpha};$$

(b) find a solution $\bar{U}$, $\bar{R}$, of the 'Signorini, μ-Coulomb, standard' problem (5.5), with the data $\mathcal{U}_{\text{loc free}} = \bar{U}^{\alpha}_{\text{loc free}}$, $\mathcal{W} = \bar{W}^{\alpha\alpha}, \mu = \mu^{\alpha}$;

(c) set $R^{\alpha}(p+1) = \bar{R}$;

(2) reactions are updated and the next candidate is called;

(3) the list of candidates is read (p is increased) until some quality criterion is satisfied or $p = p_{\max}$.

This method is similar to a nonlinear block Gauss–Seidel method.

6. Derived 'Signorini, μ-Coulomb, standard' laws

In this section the concept of 'Signorini, μ-Coulomb, derived' law, is introduced: some variables, images of the local basic variables, gap, relative velocity, reaction force, by some affine mappings, satisfy the standard Signorini relation and Coulomb's law. Such changes of variables are defined at the beginning of the time-step by some status, a data storage that is supposed to summarize the significant history of contact.

(*a*) *Derived law*

A *Signorini, μ-Coulomb, derived* law is defined by

1. *a status variable*, $S^{\alpha}_{\text{status}}(i)$, entirely defined by the relative velocity, the gap, the mean value of the impulse, $U^{\alpha}(i)$, $g^{\alpha}(i)$, $R^{\alpha}(i)$, and possibly status at previous time-steps $j < i$;

2. *auxiliary unknowns*, $\bar{U}$, $\bar{R}$, affine functions of $U^{\alpha}(i+1)$, $g^{\alpha}(i+1)$, $R^{\alpha}(i+1)$; the coefficients of these functions depend on the status $S^{\alpha}_{\text{status}}(i)$ at step i, and $U^{\alpha}(i)$, $g^{\alpha}(i)$, $R^{\alpha}(i)$;

3. according to the status, either $R^{\alpha}(i+1) = 0$, or the auxiliary variables $\bar{U}$, $\bar{R}$, satisfy 'Signorini, μ-Coulomb, standard' law, (3.5), (3.8).

(*b*) *Change of variables*

The index α is momentarily omitted. It has been seen that the relative velocity and the reaction force are related through the linearized dynamical equation (5.4),

$$U(i+1) = U_{\text{loc free}} + WhR(i+1). \tag{6.1}$$

The gap is given by the predictive formula (4.7),

$$g(i+1) = g(i) + hU_{\text{N}}(i+1). \tag{6.2}$$

The auxiliary unknowns $\bar{U}$, $\bar{R}$ are introduced as affine mappings of $U(i+1)$, $g(i+1)$, $R(i+1)$, written as

$$\begin{pmatrix} \bar{U} \\ h\bar{R} \end{pmatrix} = \begin{pmatrix} I_{UU} & I_{UR} \\ I_{RU} & I_{RR} \end{pmatrix} \begin{pmatrix} U(i+1) \\ hR(i+1) \end{pmatrix} + \begin{pmatrix} \bar{U}_0 \\ h\bar{R}_0 \end{pmatrix}. \tag{6.3}$$

It is supposed that the matrix,

$$\begin{pmatrix} I_{UU} & I_{UR} \\ I_{RU} & I_{RR} \end{pmatrix},$$

is one-to-one. An equivalent form of the previous relation is then

$$\begin{pmatrix} U(i+1) \\ hR(i+1) \end{pmatrix} = \begin{pmatrix} J_{UU} & J_{UR} \\ J_{RU} & J_{RR} \end{pmatrix} \begin{pmatrix} \bar{U} \\ h\bar{R} \end{pmatrix} + \begin{pmatrix} U_0 \\ hR_0 \end{pmatrix}. \tag{6.4}$$

Thus

$$(J_{UU} - WJ_{RU})\bar{U} = (WJ_{RR} - J_{UR})h\bar{R} + WhR_0 - U_0 + U_{\text{loc free}}.$$

It is supposed that the matrix $J_{UU} - WJ_{RU}$ is one-to-one. The previous relation becomes

$$\bar{U} = \bar{U}_{\text{loc free}} + \bar{W} h \bar{R}, \tag{6.5}$$

where

$$\begin{aligned} \bar{U}_{\text{loc free}} &= U_{\text{change}} + C_{\text{change}} U_{\text{loc free}}, \\ \bar{W} &= C_{\text{change}}(WJ_{RR} - J_{UR}), \\ U_{\text{change}} &= C_{\text{change}}(WhR_0 - U_0), \\ C_{\text{change}} &= (J_{UU} - WJ_{RU})^{-1}. \end{aligned}$$

The class of change of variables must be restricted so as to satisfy the objectivity principle as well as thermodynamic principles.

It should be noticed that the data J_{UU}, J_{UR}, J_{RU}, J_{RR}, U_0, hR_0, the matrices $\bar{W}$, C_{change}, the vector U_{change}, constructed with physical constants, depend only on $S^{\alpha}_{\text{status}}(i)$ and $U^{\alpha}(i)$, $g^{\alpha}(i)$, $R^{\alpha}(i)$. These data are to be prepared at the beginning of the time-step, before undergoing iterations. Equation (6.5) is the corresponding form of the linearized dynamical equation (6.1) through changes of variables (6.3), (6.4).

(*c*) *Solving the frictional contact problem with a derived law*

The algorithm § 6 *b* is easily generalized.

Prepare the data

$$\begin{gathered} J^{\alpha}_{UU}, \quad J^{\alpha}_{UR}, \quad J^{\alpha}_{RU}, \quad J^{\alpha}_{RR}, U^{\alpha}_{0}, \quad hR^{\alpha}_{0}, \quad C^{\alpha}_{\text{change}}, \quad U^{\alpha}_{\text{change}}, \\ \bar{W}^{\alpha\alpha} = C^{\alpha}_{\text{change}}(W^{\alpha\alpha} J^{\alpha}_{RR} - J^{\alpha}_{UR}), \end{gathered}$$

for all α and status $S^{\alpha}_{\text{status}}(i)$. The unknowns $R^{\alpha}(i+1)$ are sought as limits of the sequences $R^{\alpha}(p)$, $p = 1, \ldots, p_{\max}$:

(1) at iteration $p+1$, seek an approximate solution at candidate α; provisional values at other candidates are adopted as follows: if $\beta > \alpha$, these are the values computed at iteration p, and if $\beta < \alpha$, these are the values just computed at iteration $p+1$,

$$R^{\beta}(p+1),\ \beta = 1, \ldots, \alpha - 1, \quad R^{\beta}(p),\ \beta = \alpha + 1, \ldots, \chi;$$

compute the free velocity,

$$U^{\alpha}_{\text{loc free}} = U^{\alpha}_{\text{free}} + \sum_{\beta < \alpha} W^{\alpha\beta} h R^{\beta}(p+1) + \sum_{\beta > \alpha} W^{\alpha\beta} h R^{\beta}(p);$$

(a) according to $S^{\alpha}_{\text{status}}(i)$, compute $\bar{U}^{\alpha}_{\text{loc free}} = U^{\alpha}_{\text{change}} + C^{\alpha}_{\text{change}} U^{\alpha}_{\text{loc free}}$;

(b) find a solution $\bar{U}$, $\bar{R}$, of the 'Signorini, μ-Coulomb, standard' problem (5.5), with the data $\mathcal{U}_{\text{loc free}} = \bar{U}^{\alpha}_{\text{loc free}}$, $\mathcal{W} = \bar{W}^{\alpha\alpha}, \mu = \mu^{\alpha}$;

(c) make the reverse change of variables (6.4) to obtain $R^{\alpha}(p+1)$ from $\bar{R}$,

$$\begin{pmatrix} U^{\alpha}(i+1) \\ hR^{\alpha}(i+1) \end{pmatrix} = \begin{pmatrix} J^{\alpha}_{UU} & J^{\alpha}_{UR} \\ J^{\alpha}_{RU} & J^{\alpha}_{RR} \end{pmatrix} \begin{pmatrix} \bar{U} \\ h\bar{R} \end{pmatrix} + \begin{pmatrix} U^{\alpha}_{0} \\ hR^{\alpha}_{0} \end{pmatrix};$$

(2) reactions are updated and the next candidate is called;

(3) the list of candidates is read (p is increased) until some quality criterion is satisfied or $p = p_{\max}$.

The 'ordinary' frictional contact problem, presented as an introductory example in § 5, using (4.8), (4.9), may be considered as a special case, where a default status is assumed.

(*d*) *Non-frictional cohesive example*

As a second example, the NSCD method is applied to the non-frictional cohesive law already presented in § 1.

Prepare the data:

(1) if $S^\alpha_{\text{status}}(i)$ is 'non-cohesive', the following change of variables is adopted, i.e. the 'ordinary' unilateral, frictional model (4.8), (4.9) is used,

$$\bar{U}_{\mathrm{T}} = U^\alpha_{\mathrm{T}}(i+1), \quad \bar{U}_{\mathrm{N}} = \frac{1}{h} g^\alpha(i+1),$$
$$\bar{R} = R^\alpha(i+1),$$
$$\mu^\alpha = 0, \quad \mu^\alpha \text{ friction coefficient.}$$

It follows that

$$\bar{U}^\alpha_{\text{loc free T}} = U^\alpha_{\text{loc free T}}, \quad \bar{U}^\alpha_{\text{loc free N}} = U^\alpha_{\text{loc free N}} + \frac{1}{h} g^\alpha(i), \quad \bar{W}^{\alpha\alpha} = W^{\alpha\alpha},$$

(2) if $S^\alpha_{\text{status}}(i)$ is 'cohesive', the following change of variables is adopted,

$$\begin{aligned} &\bar{U}_{\mathrm{T}} = U^\alpha_{\mathrm{T}}(i+1), && \bar{U}_{\mathrm{N}} = \frac{1}{h} g^\alpha(i+1), \\ &\bar{R}_{\mathrm{T}} = R^\alpha_{\mathrm{T}}(i+1), && \bar{R}_{\mathrm{N}} = R^\alpha_{\mathrm{N}}(i+1) + c^\alpha, \\ &c^\alpha > 0, && c^\alpha \text{ cohesion constant}, \\ &\mu^\alpha = 0, && \mu^\alpha \text{ friction coefficient.} \end{aligned}$$

Setting

$$W^{\alpha\alpha} = \begin{pmatrix} W^{\alpha\alpha}_{\mathrm{TT}} & W^{\alpha\alpha}_{\mathrm{TN}} \\ W^{\alpha\alpha}_{\mathrm{NT}} & W^{\alpha\alpha}_{\mathrm{NN}} \end{pmatrix},$$

it follows that

$$\bar{U}^\alpha_{\text{loc free T}} = U^\alpha_{\text{loc free T}} - W^{\alpha\alpha}_{\mathrm{TN}} c^\alpha, \quad \bar{U}^\alpha_{\text{loc free N}} = U^\alpha_{\text{loc free N}} + \frac{1}{h} g^\alpha(i) - W^{\alpha\alpha}_{\mathrm{NN}} c^\alpha,$$
$$\bar{W}^{\alpha\alpha} = W^{\alpha\alpha}.$$

Some status change rules have to be chosen. The following rules are adopted:

$S^\alpha_{\text{status}}(i+1)$ is the same as $S^\alpha_{\text{status}}(i)$,
except
if $S^\alpha_{\text{status}}(i)$ is 'cohesive',
and if the solution $R^\alpha_{\mathrm{N}}(i+1)$ satisfies $R^\alpha_{\mathrm{N}}(i+1) + c^\alpha < 0$, or $g^\alpha(i+1) > 0$,
then $S^\alpha_{\text{status}}(i+1)$ is set to 'non-cohesive'.

When candidates to contact have been separated, the candidates are not allowed to be glued again. This is the rule in the next applications, §§ 7 and 8. Other rules may be adopted.

The case of *Mohr–Coulomb law* is a mere generalization of the above case. One has only to set μ at the value of the friction coefficient μ^α in the non-cohesive case, and to set $\mu = \mathrm{tg}(\varphi^\alpha)$ in the cohesive case, where φ^α is the internal friction angle. The change of status rule is

$S^\alpha_{\mathrm{status}}(i+1)$ is the same as $S^\alpha_{\mathrm{status}}(i)$,
except
if $S^\alpha_{\mathrm{status}}(i)$ is 'cohesive',
and if the solution $R^\alpha(i+1)$ is such that
$(R^\alpha_{\mathrm{T}}(i+1), R^\alpha_{\mathrm{N}}(i+1) + c^\alpha)$ lies outside the Coulomb cone with φ^α opening
or $g^\alpha(i+1) > 0$,
then $S^\alpha_{\mathrm{status}}(i+1)$ is set to 'non-cohesive'.

One sees that many laws may be constructed, playing with affine changes of variables and status change rules. Definitions of status may also be refined. This method applies as well to finite-elements models of contacting bodies that to collection of rigid bodies as granular materials. More details may be found in Cambou *et al.* (2001). As a matter of fact, if the use of (3.4), (3.7) is relevant when one of the contacting bodies is deformable, shock laws are to be used when both contacting bodies are rigid.

Clarify sentence?

(*e*) *The Frémond–Cangemi–Raous cohesive model*

The previous model, the Mohr–Coulomb model, is 'very' non-smooth. Nevertheless, it is extensively used in soil mechanics, with some refinement, for instance, the Cam clay law or the Cambou–Jafari–Sidoroff model. The concept of damage is often related to some decohesive process. Refining the time-scale, introducing damage, is a way of coming into details, and of describing more precisely the evolution of interface stresses when decohesion is occurring, i.e. going from undamaged to completely damaged interfaces. Such models are smoother than the mere Mohr–Coulomb model. Furthermore, it is possible to derive such models from thermodynamic considerations. The energy balance between the different contributions is thus easy to identify. In comparison, global energy estimations are possible with 'Signorini, μ-Coulomb, derived' laws, as it is possible for shock laws, but the way the energy is dissipated is deliberately ignored.

Clarify sentence?

The following Frémond–Cangemi–Raous (FCR) model, based on a thermodynamic approach (see Cangémi *et al.* 1996; Raous *et al.* 1997, 1999) belongs to the class of smoothed models in the above sense, i.e. a surface damage variable is introduced, together with a damage law governing the evolution of the damage variable. Strong velocity variations or discontinuities are expected during the decohesive process and dynamics should be taken into consideration. It is possible to admit the FCR model in the class of 'Signorini, μ-Coulomb, derived' laws, allowing us to use straightforwardly the implicit NSCD method. The FCR model is hereafter presented in this context. At first an intermediate model is considered. The index α is omitted for simplicity.

of what?

Prepare the data:

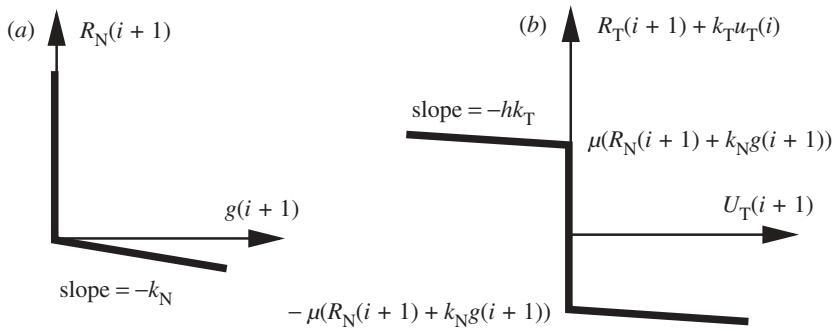

Figure 3. (a) Unilateral condition; (b) friction law.

(1) if $S_{\text{status}}(i)$ is 'non-cohesive', the following change of variables is adopted, i.e. the 'ordinary' unilateral, frictional model (4.8), (4.9) is used,

$$\bar{U}_{\mathrm{T}} = U_{\mathrm{T}}(i+1), \quad \bar{U}_{\mathrm{N}} = \frac{1}{h} g(i+1),$$
$$\bar{R} = R(i+1),$$
$$\mu \text{ is the non-cohesive status friction coefficient,}$$

(2) if $S_{\text{status}}(i)$ is 'cohesive', the following change of variables is adopted,

$$\begin{aligned} \bar{U}_{\mathrm{T}} &= U_{\mathrm{T}}(i+1), \quad \bar{U}_{\mathrm{N}} = \frac{1}{h} g(i+1), \\ \bar{R}_{\mathrm{T}} &= R_{\mathrm{T}}(i+1) - k_{\mathrm{T}} u_{\mathrm{T}}(i+1), \\ \bar{R}_{\mathrm{N}} &= R_{\mathrm{N}}(i+1) - k_{\mathrm{N}} g(i+1), \\ \mu &\text{ is the cohesive status friction coefficient.} \end{aligned} \qquad (\diamond)$$

The symbols k_{T}, k_{N} are stiffness constants (depending on α). The variable u_{T} is the 'tangential displacement', the definition of which is

$$u_{\mathrm{T}}(t) = \int_{]t_0,t]} U_{\mathrm{T}}(s)\,\mathrm{d}s,$$

where t_0 is the instant where the contact is set. Actually, the definition and the numerical estimation of the tangential displacement gives rise to serious difficulties. Within the small perturbations assumption, with finite sliding between small curvature bodies, the following approximate formula may be adopted,

$$u_{\mathrm{T}}(i+i) = u_{\mathrm{T}}(i) + hU_{\mathrm{T}}(i+1), \qquad (6.6)$$

so that the above formula quoted ($\diamond$) may be written as

$$\bar{R}_{\mathrm{T}} = R_{\mathrm{T}}(i+1) - k_{\mathrm{T}} u_{\mathrm{T}}(i) - hk_{\mathrm{T}} U_{\mathrm{T}}(i+1).$$

If the variables $\bar{U}$, $\bar{R}$, satisfy the 'Signorini, μ-Coulomb, standard' law, the variables, $R_{\mathrm{N}}(i+1)$, $(1/h)g(i+1)$ lie on figure 3a, and the variables $R_{\mathrm{T}}(i+1)$, $U_{\mathrm{T}}(i+1)$ lie on figure 3b. It means that, when the status is cohesive, some spring with stiffness k_{N} is acting from the actual antagonist point to the actual candidate point in the normal

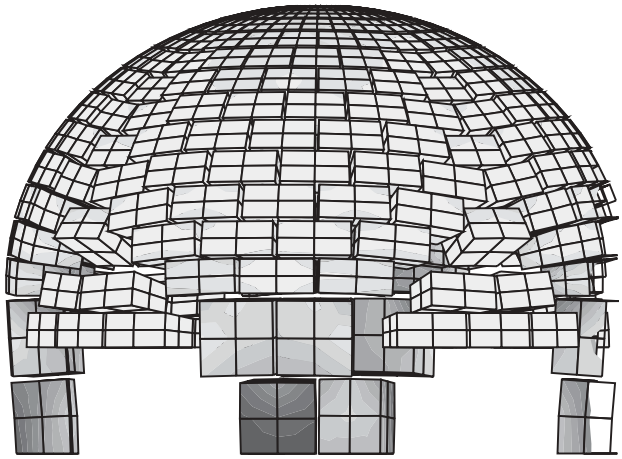

Figure 5. Non-cohesive dome.

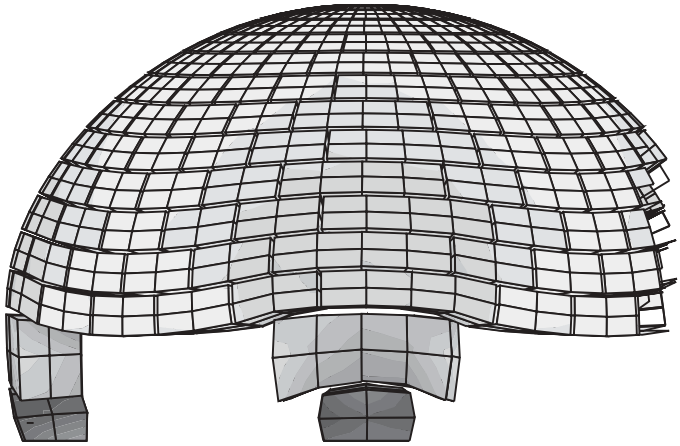

Figure 6. Cohesive dome.

are beginning to fall between pillars (displacements are magnified 2×10^4 times). In the second example, some cohesion has been introduced, as much to allow any joint to resist a tensile force equal to the weight of six blocks, enough to ensure the ability of six-block layers between pillars to resist gravity. In these circumstances the dome is stable (displacements are magnified 3×10^5 times). The scale shows the Von Mises stresses and particularly, compression in the pillars (black) and traction in layers between pillars (white). (See Acary & Jean (1998, 2000).)

9. Annex: the two-dimensional 'Signorini, μ-Coulomb, standard' problem solution

One sets

$$\mathcal{W} = \begin{pmatrix} \mathcal{W}_{\mathrm{TT}} & \mathcal{W}_{\mathrm{TN}} \\ \mathcal{W}_{\mathrm{NT}} & \mathcal{W}_{\mathrm{NN}} \end{pmatrix}, \qquad h\mathcal{R}_{\text{loc stick}} = -\mathcal{W}^{-1}\mathcal{U}_{\text{loc free}}.$$

A shorter notation is adopted, ($\mathcal{U}_{\text{free}} = \mathcal{U}_{\text{loc free}}$, $\mathcal{R}_{\text{stick}} = \mathcal{R}_{\text{loc stick}}$). It is supposed that $\mathcal{W}$ is positive definite. The term $\mathcal{U}_{\text{free}}$, ($\mathcal{U}_{\text{loc free}}$) is the free local velocity, i.e.

the local velocity when reaction is discarded on the concerned contact, $\mathcal{R}=0$; the term $\mathcal{R}_{\text{stick}}$, ($\mathcal{R}_{\text{loc stick}}$) may be viewed as the reaction when a null relative velocity is imposed, $\mathcal{U}=0$.

$$\text{It is supposed } -1<-\mu\frac{\mathcal{W}_{\text{NT}}}{\mathcal{W}_{\text{NN}}}<1, \tag{9.1}$$

The *standard solution* is

$$\left.\begin{array}{l}
\text{if } \mathcal{U}_{\text{free N}}>0, \quad \text{then: no contact,}\\
h\mathcal{R}=0;\\
\text{if } \mathcal{U}_{\text{free N}}\leqslant 0 \text{ and } \mathcal{R}_{\text{stick T}}+\mu\mathcal{R}_{\text{stick N}}<0, \quad \text{then: forward frictional sliding,}\\
h\mathcal{R}_{\text{T}}=-\mu h\mathcal{R}_{\text{N}}, \quad h\mathcal{R}_{\text{N}}=-\dfrac{1}{(1-\mu(\mathcal{W}_{\text{NT}}/\mathcal{W}_{\text{NN}}))\mathcal{W}_{\text{NN}}}\mathcal{U}_{\text{free N}};\\
\text{if } \mathcal{U}_{\text{free N}}\leqslant 0 \text{ and } \mathcal{R}_{\text{stick T}}-\mu\mathcal{R}_{\text{stick N}}>0, \quad \text{then: backward frictional sliding,}\\
h\mathcal{R}_{\text{T}}=\mu h\mathcal{R}_{\text{N}}, h\mathcal{R}_{\text{N}}=-\dfrac{1}{(1+\mu(\mathcal{W}_{\text{NT}}/\mathcal{W}_{\text{NN}}))\mathcal{W}_{\text{NN}}}\mathcal{U}_{\text{free N}};\\
\text{if } \mathcal{U}_{\text{free N}}\leqslant 0 \text{ and } \mathcal{R}_{\text{stick T}}+\mu\mathcal{R}_{\text{stick N}}\geqslant 0 \text{ and } \mathcal{R}_{\text{stick T}}-\mu\mathcal{R}_{\text{stick N}}\leqslant 0,\\
\qquad\qquad \text{then: sticking,}\\
h\mathcal{R}=h\mathcal{R}_{\text{stick}}.
\end{array}\right\} \tag{9.2}$$

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