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Necessary conditions of asymptotic stability for unilateral dynamical systems

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Abstract

In this paper, we develop a mathematical tool that can be used to state necessary conditions of asymptotic stability of isolated stationary solutions of a class of unilateral dynamical systems. More precisely, nonlinear evolution variational inequalities are considered. Instability criteria are also given. Applications can be found in mechanics or electrical circuits.

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1. Introduction

The stability of stationary solutions of dynamical systems constitutes a very important topic in Applied Mathematics and Engineering. It is well-known that in the case of a large class of nonlinear differential equations, the spectrum of "linearized" operators determines the Lyapunov stability or instability of an equilibrium. This is known as the Lyapunov's linearization method [28,27,34]. However, many important problems in engineering (see [4,10,11,14,13,17,29,33]) involve inequalities in their mathematical formulation and

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consequently possess intrinsic nonsmoothness which cannot be removed by change of coordinates, or by feedback. The models corresponding to such problems are called unilateral dynamical systems or nonregular dynamical systems. A large class of unilateral dynamical systems can be represented under the formalism of evolution variational inequalities, see e.g. [3,14,13]. The question of stability of stationary solutions of evolution variational inequalities is much more complicated to be investigated, as it is the case in general for hybrid dynamical systems, see e.g. [20,21].

The research efforts to develop general mathematical approaches to study stability of stationary solutions of evolution variational inequalities are relatively new. It seems that the first contribution acting in this sense was made by Quittner [23,25,24] for a class of parabolic variational inequalities in Hilbert spaces. More recently, Lyapunov approach and Krakovskii–LaSalle invariance theory have been generalized to evolution variational inequalities and corresponding unilateral dynamical systems (see [1,12,6,8,15,22] and the references cited therein). In [12] sufficient conditions for stability of linear evolution variational inequalities have been derived, and we now propose to study necessary conditions.

Our aim in this paper is to state necessary conditions of asymptotic stability for a class of unilateral dynamical systems. More precisely, we consider the problem: Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous function. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a nonlinear continuous operator. Let $x_0 \in \mathbb{R}^n$ be given. We consider the problem $P(x_0)$: Find a unique continuous mapping $t \mapsto u(t)$ such that

$$\frac{\mathrm{d}u}{\mathrm{d}t} \in L^{\infty}_{\mathrm{loc}}(0, +\infty; \mathbb{R}^n), \tag{1}$$

$$u(t) \in D(\partial\varphi), \quad \forall t \geqslant 0, \tag{2}$$

$$\left\langle \frac{\mathrm{d}u}{\mathrm{d}t}(t) + F(u(t)), v - u(t) \right\rangle + \varphi(v) - \varphi(u(t)) \geqslant 0, \quad \forall v \in \mathbb{R}^n, \quad \text{a.e. } t \geqslant 0 \tag{3}$$

and

$$u(0) = x_0. \tag{4}$$

The variational inequality in (3) can also be formulated as the set-valued differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + F(u(t)) \in -\partial\varphi(u(t)), \quad \text{a.e. } t \geqslant 0, \tag{5}$$

where $\partial\varphi$ denotes the subdifferential of φ. It is assumed that $F(0) \in -\partial\varphi(0)$ i.e. the origin 0 of the system is a trivial stationary solution of (5).

If $\varphi \equiv 0$ then (5) reduces to a standard system of differential equations and it is well known (see e.g. [19]) in this case that if 0 is an isolated zero of F and is asymptotically stable, then there exists $\rho_0 > 0$ such that

$$\deg(F, B_\rho, 0) = 1, \quad \forall \rho \in (0, \rho_0],$$

where $\deg(F, B_\rho, 0)$ denotes the Brouwer degree of F with respect to the open ball $B_\rho := \{x \in \mathbb{R}^n : \|x\| < \rho\}$ and 0.

Our goal in this paper is to generalize this famous result to the model in (4) and (5). More precisely, let us define the mapping Λ as

$$\Lambda(x) := x - P_\varphi(x - F(x)),$$

where $P_\varphi := (\mathrm{id}_{\mathbb{R}^n} + \partial\varphi)^{-1}$ and $\mathrm{id}_{\mathbb{R}^n}$ denotes the identity mapping on $\mathbb{R}^n$. We will prove that if 0 is an isolated zero of Λ and is asymptotically stable, then there exists $\rho_0 > 0$ such that

$$\deg(\Lambda, B_\rho, 0) = 1, \quad \forall \rho \in (0, \rho_0].$$

This result constitutes the main result of the paper and is given in Theorem 5.

In Section 2, we recall the fundamental properties of the Brouwer degree we will use in this paper. In Section 3, we discuss the main properties of the operator P_φ. In Section 4, we present a concise review of some recent results in stability theory of unilateral dynamical systems. In Section 5, we introduce the Poincaré operator associated to problem (5). In Section 6, we prove our main result.

In this paper, we develop also several approaches to compute the number $\deg(\Lambda, B_\rho, 0)$. If Λ can be easily evaluated then a first approach consist to use some practical results of degree theory to compute $\deg(\Lambda, B_\rho, 0)$ (see Section 6). The approach is of particular interest if $\varphi \equiv \Psi_{\mathbb{R}^n_+}$, with $\Psi_{\mathbb{R}^n_+}$ denoting the indicator function of $\mathbb{R}^n_+$, since in this case:

$$\Lambda_i(x) = \min\{x_i, F_i(x)\}.$$

This case is of particular interest for the study of complementarity dynamical systems [16].

More generally, if $\varphi \equiv \Psi_K$ where K denotes a closed convex set, then we develop a second approach (see Section 7) that uses a result of Quittner [24] which reduces the computation of $\deg(\Lambda, B_\rho, 0)$ to the one of $\deg(\Lambda_L, B_\rho, 0)$ where Λ_L is defined by

$$\Lambda_L(x) = x - P_{K_0}(x - J_F(0)x),$$

where K_0 is the closed convex cone defined by $K_0 = \bigcup_{\alpha>0} \alpha K$ and $J_F(0)$ denotes the Jacobian matrix of F at 0.

If φ is a convex and continuous function then we develop a third approach (see Section 8) by showing that $\deg(\Lambda, B_\rho, 0)$ can be computed by using some appropriate "Lyapunov function".

The case of linear complementarity problems is discussed in Section 9. Finally, in Section 10, using our main result together with recent invariant results (see [1]), we prove some instability theorems.

Several examples and applications are given so as to illustrate the theoretical results.

Notations. In the sequel the scalar product on $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$ (with the associated norm $\| \cdot \|$). For $r > 0$, we set $B_r := \{x \in \mathbb{R}^n : \|x\| < r\}$. Then $\bar{B}_r = \{x \in \mathbb{R}^n : \|x\| \leqslant r\}$ and $\partial B_r = \{x \in \mathbb{R}^n : \|x\| = r\}$.

The identity mapping on $\mathbb{R}^n$ is denoted by $\mathrm{id}_{\mathbb{R}^n}$. Let K be a nonempty closed convex subset of $\mathbb{R}^n$. The dual set of K is defined by $K^* := \{w \in \mathbb{R}^n : \langle w, x\rangle \geqslant 0, \forall x \in K\}$. The recession cone of K is defined by $K_\infty := \bigcup_{\lambda>0}(1/\lambda)(K - x_0)$ where x_0 is any element of K. We denote by ψ_K the indicator function of K, i.e. $\psi_K(x) = 0$ if $x \in K$ and $\psi_K(x) = +\infty$ if $x \notin K$.

For a function $V \in C^1(\mathbb{R}^n; \mathbb{R})$ we denote by $V'(x)$ the gradient of V at $x \in \mathbb{R}^n$.

We denote by $d(s, \mathscr{M})$ the distance from a point $s \in \mathbb{R}^n$ to a set $\mathscr{M} \subset \mathbb{R}^n$, that is $d(s, \mathscr{M}) := \inf_{m\in\mathscr{M}}\|s - m\|$.

2. Preliminaries on topological degree theory

Let $D \subset \mathbb{R}^n$ be an open and bounded set. Let $f \in C^1(D; \mathbb{R}^n) \cap C^0(\bar{D}; \mathbb{R}^n)$ be given. Set $A_f(D) := \{x \in \bar{D} : f(x) = 0\}$ and $B_f(D) := \{x \in D : \det J_f(x) = 0\}$ where $J_f(x)$ denotes the Jacobian Matrix of f at x defined by $(J_f)_{ij} := (\partial f_i/\partial x_j)(x), i, j \in \{1, \ldots, n\}$. If $A_f(D) \cap B_f(D) = \emptyset$ and $0 \notin f(\partial D)$ then $A_f(D)$ is a finite set and the Brouwer topological degree of f with respect to D and 0 is well-defined by the formula

$$\deg(f, D, 0) = \sum_{x\in A_f(D)} \mathrm{sgn}(\det\ J_f(x)),$$

where $\mathrm{sgn}(t) = 1$ for $t > 0$ and $\mathrm{sgn}(t) = -1$ for $t < 0$. More generally, if $f : \bar{D} \to \mathbb{R}^n$ is continuous and $0 \notin f(\partial D)$ then the Brouwer topological degree of f with respect to D and 0 is well-defined (see e.g. [18]) and denoted by $\deg(f, D, 0)$.

Let us now recall some properties of the topological degree we will use later in this paper.

1. If $0 \notin f(\partial D)$ and $\deg(f, D, 0) \neq 0$ then there exists $x \in D$ such that $f(x) = 0$.
2. Let $\varphi : [0, 1] \times \bar{D} \to \mathbb{R}^n; (\lambda, x) \mapsto \varphi(\lambda, x)$, be continuous such that, for each $\lambda \in [0, 1]$, one has $0 \notin \varphi(\lambda, \partial D)$. Then the map $\lambda \mapsto \deg(\varphi(\lambda, .), D, 0)$ is constant on $[0, 1]$.
3. If $0 \in D$ then
$$\deg(\mathrm{id}_{\mathbb{R}^n}, D, 0) = 1.$$
4. If $0 \notin f(\partial D)$ and $\alpha > 0$ then
$$\deg(\alpha f, D, 0) = \deg(f, D, 0)$$
and
$$\deg(-\alpha f, D, 0) = (-1)^n \deg(f, D, 0).$$
5. Let $D' \subset D$ be an open set such that $A_f(D) \subset D'$. Then
$$\deg(f, D, 0) = \deg(f, D', 0).$$
6. Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. If $0 \in D$ then $\deg(A., D, 0) = \mathrm{sgn}(\det\ A)$.

7. Let $f : [-r, +r] \to \mathbb{R}$ be a continuous function. Suppose that $f(-r) \neq 0$ and $f(+r) \neq 0$. Then

$$\deg(f,]-r, +r[, 0) = 0 \quad \text{if } f(-r)f(+r) > 0,$$

$$\deg(f,]-r, +r[, 0) = +1 \quad \text{if } f(-r) < 0 \text{ and } f(+r) > 0,$$

$$\deg(f,]-r, +r[, 0) = -1 \quad \text{if } f(-r) > 0 \text{ and } f(+r) < 0.$$

3. Generalized projection operator and its inverse

Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a proper, convex and lower semi-continuous mapping. We denote by $\mathrm{dom}\{\varphi\}$ the domain of φ, i.e.

$$\mathrm{dom}\{\varphi\} := \{x \in \mathbb{R}^n : \varphi(x) < +\infty\}.$$

We denote by $\partial\varphi$ the convex subdifferential of φ. Recall that

$$\partial\varphi(x) = \{w \in \mathbb{R}^n : \varphi(v) - \varphi(x) \geqslant \langle w, v - x\rangle, \forall v \in \mathbb{R}^n\}.$$

We denote by $D(\partial\varphi)$ the domain of $\partial\varphi$ and by $R(\partial\varphi)$ the range of $\partial\varphi$, i.e.

$$D(\partial\varphi) := \{x \in \mathbb{R}^n : \partial\varphi(x) \neq \emptyset\}, \quad R(\partial\varphi) := \bigcup_{x\in\mathbb{R}^n} \partial\varphi(x).$$

Recall that

$$D(\partial\varphi) \subset \mathrm{dom}\{\varphi\} \subset \overline{D(\partial\varphi)}. \tag{6}$$

Let $y \in \mathbb{R}^n$ be given. We consider the variational inequality problem: Find $x \in \mathbb{R}^n$ such that

$$\langle x - y, v - x\rangle + \varphi(v) - \varphi(x) \geqslant 0, \quad \forall v \in \mathbb{R}^n. \tag{7}$$

Problem (7) has a unique solution (see e.g. [14,30]) that we denote by $P_\varphi(y)$. The operator

$$P_\varphi : \mathbb{R}^n \to \mathbb{R}^n;\ y \mapsto P_\varphi(y)$$

is thus well-defined. It is clear that

$$P_\varphi(\mathbb{R}^n) \subset D(\partial\varphi) \tag{8}$$

and

$$P_\varphi(y) = (\mathrm{id}_{\mathbb{R}^n} + \partial\varphi)^{-1}(y).$$

If $\varphi \equiv \psi_K$ where K is a nonempty closed convex set and ψ_K denotes the indicator function of K, then

$$P_\varphi \equiv P_{\psi_K} \equiv P_K,$$

where P_K denotes the projection operator onto K which is defined by the formula

$$\|x - P_K x\| = \min_{w \in K} \|x - w\|.$$

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and consider the inequality problem: Find $\bar{x} \in \mathbb{R}^n$ such that

$$\langle \Phi(\bar{x}), v - \bar{x} \rangle + \varphi(v) - \varphi(\bar{x}) \geqslant 0, \quad \forall v \in \mathbb{R}^n. \tag{9}$$

It is clear that problem (9) is equivalent to the nonlinear equation: Find $\bar{x} \in \mathbb{R}^n$ such that

$$\bar{x} - P_\varphi(\bar{x} - \Phi(\bar{x})) = 0. \tag{10}$$

Proposition 1. *The operator P_φ is nonexpansive, i.e.*

$$\|P_\varphi v - P_\varphi y\| \leqslant \|v - y\|, \quad \forall v, y \in \mathbb{R}^n.$$

Proof. Let $v, y \in \mathbb{R}^n$ be given. We set $x := P_\varphi(v)$ and $x^* := P_\varphi(y)$. We have

$$\langle x - v, w - x \rangle + \varphi(w) - \varphi(x) \geqslant 0, \quad \forall w \in \mathbb{R}^n \tag{11}$$

and

$$\langle x^* - y, w - x^* \rangle + \varphi(w) - \varphi(x^*) \geqslant 0, \quad \forall w \in \mathbb{R}^n. \tag{12}$$

Setting $w := x^*$ in (11) and $w := x$ in (12), we obtain the relations $\langle x - v, x - x^* \rangle - \varphi(x^*) + \varphi(x) \leqslant 0$ and $-\langle x^* - y, x - x^* \rangle - \varphi(x) + \varphi(x^*) \leqslant 0$, from which we deduce that $\|x - x^*\|^2 \leqslant \|x - x^*\| \|v - y\|$. It follows that

$$\|P_\varphi(v) - P_\varphi(y)\| \leqslant \|v - y\|. \quad \square$$

Let us now denote by $\mathscr{A}_\varphi : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ the set-valued operator defined by

$$\mathscr{A}_\varphi(x) = \{f \in \mathbb{R}^n : \langle x - f, v - x \rangle + \varphi(v) - \varphi(x) \geqslant 0, \forall v \in \mathbb{R}^n\}. \tag{13}$$

We see that

$$\mathscr{A}_\varphi^{-1}(f) = P_\varphi(f). \tag{14}$$

It is also easy to see that

$$(t\mathscr{A}_\varphi)^{-1}(tf) = P_\varphi(f), \quad \forall t > 0. \tag{15}$$

Note that

$$\mathscr{A}_\varphi(x) = \partial \left\{ \frac{\|.\|^2}{2} + \varphi(.) \right\}(x), \quad \forall x \in \mathbb{R}^n,$$

so that $\mathscr{A}_\varphi$ is maximal monotone. It results that for any $t > 0$, the operator $(\mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1}$ is a well-defined single-valued operator.

4. Unilateral dynamical systems

Let us first recall a general existence and uniqueness of solutions result (see e.g. [15]).

Theorem 1. *Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous function. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous operator such that for some $\bar{\omega} \in \mathbb{R}$, $F + \bar{\omega} I$ is monotone. Let $x_0 \in D(\partial\varphi)$ be given. There exists a unique $u \in C^0([0, +\infty); \mathbb{R}^n)$ such that*

$$\frac{du}{dt} \in L^\infty_{loc}(0, +\infty; \mathbb{R}^n), \tag{16}$$

$$u \text{ is right-differentiable on } [0, +\infty), \tag{17}$$

$$u(t) \in D(\partial\varphi), \quad \forall t \geqslant 0, \tag{18}$$

$$\left\langle \frac{du}{dt}(t) + F(u(t)), v - u(t) \right\rangle + \varphi(v) - \varphi(u(t)) \geqslant 0, \quad \forall v \in \mathbb{R}^n, \text{ a.e. } t \geqslant 0, \tag{19}$$

$$u(0) = x_0. \tag{20}$$

Remark 1. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ can be written as

$$F(x) = Ax + \Phi'(x) + F_1(x), \quad \forall x \in \mathbb{R}^n,$$

where $A \in \mathbb{R}^{n\times n}$ is a real matrix, $\Phi \in C^1(\mathbb{R}^n; \mathbb{R})$ is convex and F_1 is Lipschitz continuous, i.e.

$$\|F_1(x) - F_1(y)\| \leqslant k\|x - y\|, \quad \forall x, y \in \mathbb{R}^n,$$

for some constant $k > 0$. Then F is continuous and $F + \bar{\omega} I$ is monotone provided that

$$\bar{\omega} \geqslant \sup_{\|x\|=1} \langle -Ax, x\rangle + k.$$

The variational inequality in (19) can also be written as the differential inclusion

$$\frac{du}{dt}(t) + F(u(t)) \in -\partial\varphi(u(t)), \quad \text{a.e. } t \geqslant 0. \tag{21}$$

Suppose that the assumptions of Theorem 1 are satisfied and denote by $u(.; x_0)$ the unique solution of Problem $P(x_0)$ (see (1)–(4)). Let us here also denote by $\mathcal{S}(F, \varphi)$ the set of stationary solutions of (19), i.e.

$$\mathcal{S}(F, \varphi) = \{\bar{x} \in \mathbb{R}^n : \langle F(\bar{x}), w - \bar{x}\rangle + \varphi(w) - \varphi(\bar{x}) \geqslant 0, \forall w \in \mathbb{R}^n\}.$$

Note that

$$\mathcal{S}(F, \varphi) = \{\bar{x} \in \mathbb{R}^n : \bar{x} = P_\varphi(\bar{x} - F(\bar{x})\}.$$

For $x_0 \in D(\partial\varphi)$, we denote by $\gamma(x_0)$ the *orbit*

$$\gamma(x_0) := \{u(\tau; x_0); \tau \geqslant 0\}.$$

Let us now assume, without loss of generality, that the trivial solution 0 is a stationary solution of (19), i.e.

$$F(0) \in -\hat{\partial}\varphi(0). \tag{22}$$

Then

$$u(.;0) \equiv 0.$$

We may now define as in [15] the stability of the trivial solution. The stationary solution 0 is called stable if small perturbations of the initial condition $u(t_0) = 0$ lead to solutions which remain in the neighborhood of 0 for all $t \geqslant 0$, precisely:

Definition 1. The equilibrium point $x = 0$ is said to be stable in the sense of Lyapunov if for every $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon) > 0$ such that for any $x_0 \in D(\hat{\partial}\varphi)$ with $\|x_0\| \leqslant \eta$ the solution $u(\cdot; x_0)$ of problem $P(x_0)$ satisfies $\|u(t; x_0)\| \leqslant \varepsilon, \ \forall t \geqslant 0$.

If in addition the trajectories of the perturbed solutions are attracted by 0 then we say that the stationary solution is asymptotically stable, precisely:

Definition 2. The equilibrium point $x = 0$ is asymptotically stable if (1) it is stable and (2) there exists $\delta > 0$ such that for any $x_0 \in D(\hat{\partial}\varphi)$ with $\|x_0\| \leqslant \delta$ the solution $u(\cdot; x_0)$ of problem $P(x_0)$ fulfills

$$\lim_{t \to +\infty} \|u(t; x_0)\| = 0.$$

Definition 3. The equilibrium point $x = 0$ is unstable if it is not stable.

The following result generalizes to unilateral systems the Lyapunov's direct method. The approach makes an essential use of some auxiliary function $V \in C^1(\mathbb{R}^n; \mathbb{R})$.

Theorem 2 (*Goeleven et al. [15]*). *Suppose that the assumptions of Theorem* 1 *together with condition* (22) *hold. Suppose that there exists* $\sigma > 0$ *and* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1) $V(0) = 0$;
(2) $V(x) > 0, \forall x \in \overline{D(\hat{\partial}\varphi)} \cap \overline{B}_\sigma, x \neq 0$;
(3) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0, \ \forall x \in D(\hat{\partial}\varphi) \cap \overline{B}_\sigma$.

Then the trivial solution of (16)–(19) *is stable.*

Remark 2. (i) Assumptions (1) and (2) in Theorem 2 ensure the existence of a mapping $a : [0, \sigma] \to \mathbb{R}$ satisfying $a(0) = 0$ and $a(t) > 0, \forall t \in (0, \sigma]$ and such that $V(x) \geqslant a(\|x\|), \forall x \in \overline{D(\hat{\partial}\varphi)} \cap \bar{B}_\sigma$. The mapping $a(.)$ defined by $a(r) = \inf\{V(x) : r \leqslant \|x\| \leqslant \sigma\}$ $(r \geqslant 0)$ is suitable.

(ii) Assumption (3) in Theorem 2 implies that $\langle F(x), V'(x)\rangle + \langle w, V'(x)\rangle \geqslant 0, \forall w \in \hat{\partial}\varphi(x), \forall x \in D(\hat{\partial}\varphi) \cap \bar{B}_\sigma$.

(iii) If K denotes a nonempty closed convex set and $\varphi \equiv \psi_K$ then assumption (3) in Theorem 2 implies that $\langle F(x), V'(x)\rangle \geqslant 0, \forall x \in K \cap \bar{B}_\sigma$ and $x - V'(x) \in K, \forall x \in K \cap \bar{B}_\sigma$.

This last expression ensures that $-V'(x) \in T_K(x), \forall x \in \partial K \cap \bar{B}_\sigma$ which characterizes the level sets of V with respect to the boundary of K.

(iv) A more general version of Theorem 2 is given in [15].

Let $V \in C^1(\mathbb{R}^n; \mathbb{R})$ be given. We set

$$\mathscr{E}(F, \varphi, V) := \{x \in D(\partial\varphi) : \langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) = 0\}. \quad (23)$$

Recall here that a set $\mathscr{D} \subset D(\partial\varphi)$ is *invariant* provided that

$$x_0 \in \mathscr{D} \Rightarrow \gamma(x_0) \subset \mathscr{D}.$$

We recall a recent result that generalizes to unilateral systems the famous Krakovskii–LaSalle invariance principle.

Theorem 3 (*Invariance Theorem, Adly and Goeleven [1]*). *Suppose that the assumptions of Theorem* 1 *hold. Let* $\Psi \subset \mathbb{R}^n$ *be a compact set and* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *a function such that*

(1) $\varphi(.) - \varphi(. - V'(.))$ *is lower semicontinuous on* $D(\partial\varphi) \cap \Psi$;
(2) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0, \ \forall x \in D(\partial\varphi) \cap \Psi$;
(3) $D(\partial\varphi)$ is closed.

We denote by $\mathscr{M}(F, \varphi, V, \Psi)$ *the largest invariant subset of* $\mathscr{E}(F, \varphi, V) \cap \Psi$. *Then for each* $x_0 \in D(\partial\varphi)$ *such that* $\gamma(x_0) \subset \Psi$, *we have*

$$\lim_{\tau \to +\infty} d(u(\tau; x_0), \mathscr{M}(F, \varphi, V, \Psi)) = 0.$$

5. The Poincaré operator

Let $T > 0$ be given. Theorem 1 enables us to define the one parameter family $\{S(t) : 0 \leqslant t \leqslant T\}$ of operators from $D(\partial\varphi)$ into $\mathbb{R}^n$, as follows:

$$\forall y \in D(\partial\varphi), \quad S(t)y = x(t; y), \quad (24)$$

$x(.; y)$ being the unique solution on $[0, T]$ of the evolution problem $P(y)$. Note that

$$\forall y \in D(\partial\varphi), \quad S(0)y = y.$$

According to (24), the unique solution of problem (16)–(19) satisfies, in addition, the periodicity condition

$$u(0) = u(T)$$

if and only if y is a fixed point of $S(T)$, that is

$$S(T)y = y.$$

Thus the problem of the existence of a periodic solution for the evolution problem (16)–(19) is reduced to that of the existence of a fixed point for $S(T)$. The operator $S(T)$ is called "Poincaré Operator".

Lemma 1 (*See e.g., Showalter [30]*). *Let* $0 \leqslant t_1 < t_2 < +\infty$ *be given and let* $a, b \in L^1(t_1, t_2; \mathbb{R})$ *with* $b(t) \geqslant 0$ *a.e.* $t \in [t_1, t_2]$. *Let the absolutely continuous function* $w : [t_1, t_2] \to \mathbb{R}_+$ *satisfy*

$$(1-\alpha)\frac{\mathrm{d}w}{\mathrm{d}t}(t) \leqslant a(t)w(t) + b(t)w^{\alpha}(t), \quad \text{a.e. } t \in [t_1, t_2],$$

where $0 \leqslant \alpha < 1$. *Then*

$$w^{1-\alpha}(t) \leqslant w^{1-\alpha}(t_1)\mathrm{e}^{\int_{t_1}^{t} a(s)\,\mathrm{d}s} + \int_{t_1}^{t} \mathrm{e}^{\int_{s}^{t} a(q)\,\mathrm{d}q} b(s)\,\mathrm{d}s, \quad \forall t \in [t_1, t_2].$$

Theorem 4. *Suppose that the assumptions of Theorem* 1 *hold. Then for* $0 \leqslant t_1 < t_2 < +\infty$, $y, z \in D(\partial\varphi)$, *we have*

$$\|S(t_2)y - S(t_2)z\| \leqslant \sqrt{\mathrm{e}^{2\bar{\omega}(t_2-t_1)}}\,\|S(t_1)y - S(t_1)z\|.$$

Proof. Let $0 \leqslant t_1 < t_2 < +\infty$, $y, z \in D(\partial\varphi)$ be given. We know that

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}S(t)y + F(S(t)y),\, v - S(t)y \right\rangle + \varphi(v) - \varphi(S(t)y) \geqslant 0, \quad \forall v \in \mathbb{R}^n, \ \text{a.e. } t \geqslant 0 \tag{25}$$

and

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}S(t)z + F(S(t)z),\, h - S(t)z \right\rangle + \varphi(h) - \varphi(S(t)z), \quad \forall h \in \mathbb{R}^n, \ \text{a.e. } t \geqslant 0. \tag{26}$$

Setting $v = S(t)z$ in (25) and $h = S(t)y$ in (26), we obtain the relations

$$-\left\langle \frac{\mathrm{d}}{\mathrm{d}t}S(t)y + F(S(t)y),\, S(t)z - S(t)y \right\rangle - \varphi(S(t)z) + \varphi(S(t)y) \leqslant 0, \quad \text{a.e. } t \geqslant 0$$

and

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}S(t)z + F(S(t)z),\, S(t)z - S(t)y \right\rangle - \varphi(S(t)y) + \varphi(S(t)z) \leqslant 0, \quad \text{a.e. } t \geqslant 0.$$

It results that

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(S(t)z - S(t)y),\, S(t)z - S(t)y \right\rangle \leqslant \bar{\omega}\|S(t)z - S(t)y\|^2 - \langle [F + \bar{\omega}I](S(t)z) - [F + \bar{\omega}I](S(t)y),\, S(t)z - S(t)y \rangle \quad \text{a.e. } t \geqslant 0.$$

Our hypothesis ensure that $F + \bar{\omega}I$ is monotone. It results that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|S(t)z - S(t)y\|^2 \leqslant 2\bar{\omega}\|S(t)z - S(t)y\|^2, \quad \text{a.e. } t \geqslant 0. \tag{27}$$

Using Lemma 1 with $w(.) := \|S(.)z - S(.)y\|^2$, $a(.) := 2\bar{\omega}$, $b(.) = 0$, $\alpha = 0$, we get

$$\|S(t_2)z - S(t_2)y\|^2 \leqslant \|S(t_1)z - S(t_1)y\|^2 e^{2\bar{\omega}(t_2-t_1)}.$$

The conclusion follows. □

A consequence of Theorem 4 is that

$$\|S(t)y - S(t)z\| \leqslant \sqrt{e^{2\bar{\omega}t}}\|y - z\|, \quad \forall y, z \in D(\partial\varphi),\ t \in [0, T].$$

The Poincaré operator $S(T) : \mathbb{R}^n \to \mathbb{R}^n; y \to S(T)y$ is thus Lipschitz continuous on $D(\partial\varphi)$, i.e.

$$\|S(T)y - S(T)z\| \leqslant \sqrt{e^{2\bar{\omega}T}}\|y - z\|, \quad \forall y, z \in D(\partial\varphi). \tag{28}$$

Remark 3. (i) Note that if F is continuous and monotone then (28) holds with $\bar{\omega} = 0$. In this case, the Poincaré operator $S(T)$ is nonexpansive on $D(\partial\varphi)$.

(ii) If F is continuous and strongly monotone, i.e. there exists $\alpha > 0$ such that

$$\langle F(x) - F(y), x - y\rangle \geqslant \alpha\|x - y\|^2, \quad \forall x, y \in \mathbb{R}^n$$

then (28) holds with $\bar{\omega} = -\alpha < 0$ and the Poincaré operator $S(T)$ is a contraction on $D(\partial\varphi)$.

6. Necessary conditions of asymptotic stability

We suppose that:

(h_1) $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semi-continuous;
(h_2) $D(\partial\varphi)$ is closed;
(h_3) $0 \in D(\partial\varphi)$;
(h_4) There exists a neighborhood $\mathcal{N}$ of 0 and a constant $C_1 > 0$ such that

$$|\varphi(x_1) - \varphi(x_2)| \leqslant C_1\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathcal{N} \cap \mathrm{dom}\{\varphi\}.$$

Condition (h_2) together with (6) ensure that $D(\partial\varphi) = \mathrm{dom}\{\varphi\}$. Thus $D(\partial\varphi)$ is also convex. The projection operator $P_{D(\partial\varphi)}$ onto $D(\partial\varphi)$ is well-defined. Recall that

$$P^2_{D(\partial\varphi)} = P_{D(\partial\varphi)},$$

$$\|P_{D(\partial\varphi)}x - P_{D(\partial\varphi)}y\| \leqslant \|x - y\|, \quad \forall x, y \in \mathbb{R}^n$$

and

$$\langle P_{D(\partial\varphi)}x - x, w - P_{D(\partial\varphi)}x\rangle \geqslant 0, \quad \forall w \in D(\partial\varphi).$$

We suppose also that

(h_5) $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous operator such that for some $\bar{\omega} \in \mathbb{R}$, $F + \bar{\omega}I$ is monotone;
(h_6) F is locally Lipschitz at 0.

Finally, we suppose that

(h_7) 0 is an isolated stationary solution of (19), i.e.

$$0 = P_\varphi(-F(0)) \tag{29}$$

and there is a neighborhood $\mathscr{V}$ of 0 such that

$$x \neq P_\varphi(x - F(x)), \quad \forall x \in \mathscr{V}\backslash\{0\}. \tag{30}$$

Note that the relation in (29) can also be written as

$$\langle F(0), w\rangle + \varphi(w) - \varphi(0) \geqslant 0, \quad \forall w \in \mathbb{R}^n.$$

Let us now define the continuous mapping $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Lambda(x) := x - P_\varphi(x - F(x)). \tag{31}$$

If (h_7) is satisfied then for $r > 0$ small enough, $\deg(\Lambda(.), B_r, 0)$ is well defined and constant. We set

$$\zeta(\Lambda, 0) := \deg(\Lambda(.), B_r, 0) \quad \text{for } r > 0 \text{ small enough.} \tag{32}$$

Remark 4. If $n = 1$ then $\zeta(\Lambda, 0)$ can be computed by using property 7 (see Section 2) of degree. More precisely,

$$\zeta(\Lambda, 0) = 0 \quad \text{if } \Lambda(-r)\Lambda(+r) > 0, \text{ for } r > 0 \text{ small enough,}$$

$$\zeta(\Lambda, 0) = +1 \quad \text{if } \Lambda(-r) < 0 \text{ and } \Lambda(+r) > 0, \text{ for } r > 0 \text{ small enough,}$$

$$\zeta(\Lambda, 0) = -1 \quad \text{if } \Lambda(-r) > 0 \text{ and } \Lambda(+r) < 0, \text{ for } r > 0 \text{ small enough.}$$

Example 1. (i) Let F and φ be defined by $F(x) = 2x$ and $\varphi(x) = \Psi_{\mathbb{R}_+}$. Here

$$\Lambda(x) = x - \max\{0, -x\}.$$

For $r > 0$, we have $\Lambda(-r) = -2r < 0$ and $\Lambda(r) = r > 0$ and thus $\zeta(\Lambda, 0) = +1$.

(ii) Let F and φ be defined by $F(x) = -2x$ and $\varphi(x) = \Psi_{\mathbb{R}_+}$. Here

$$\Lambda(x) = x - \max\{0, 3x\}.$$

For $r > 0$, we have $\Lambda(-r) = -r < 0$ and $\Lambda(r) = -2r < 0$ and thus $\zeta(\Lambda, 0) = 0$.

More generally, let us define for $\varepsilon > 0$ the sets

$$P_i(-\varepsilon) := \{x \in [-\varepsilon, +\varepsilon]^n : x_i = -\varepsilon\}, \quad i \in \{1, \ldots, n\}$$

and

$$P_i(+\varepsilon) := \{x \in [-\varepsilon, +\varepsilon]^n : x_i = +\varepsilon\}, \quad i \in \{1, \ldots, n\}.$$

Note that

$$\bigcup_{i=1}^{n} (P_i(-\varepsilon) \cup P_i(+\varepsilon)) = \hat{\partial}(]-\varepsilon, +\varepsilon[^n).$$

Proposition 2. *Suppose that there exists* $\varepsilon_0 > 0$ *such that*

$$\Lambda_i(x)\Lambda_i(y) < 0, \quad \forall x \in P_i(-\varepsilon),\ y \in P_i(+\varepsilon),\ \varepsilon \in (0, \varepsilon_0],\ (i \in \{1, \ldots, n\}). \tag{33}$$

Then

$$\zeta(\Lambda, 0) = \prod_{i=1}^{n} \sigma_i,$$

where $\sigma_i := \operatorname{sgn}(\Lambda_i(\bar{x}^i))$ *with* $\bar{x}^i$ *denoting any element of* $P_i(+\bar{\varepsilon})$ *and* $\bar{\varepsilon}$ *any real in* $(0, +\varepsilon_0]$.

Proof. Let $0 < \varepsilon \leqslant \varepsilon_0$ be given. Assumption (33) together with the continuity of Λ_i ensure that $\operatorname{sgn}(\Lambda_i(.))$ is constant on $P_i(+\varepsilon)$. Let x_ε^i be any element of $P_i(+\varepsilon)$, we get (see Theorem (c) in [19]):

$$\deg(\Lambda(.),]-\varepsilon, +\varepsilon[^n, 0) = \prod_{i=1}^{n} \operatorname{sgn}(\Lambda_i(x_\varepsilon^i)).$$

Moreover condition (33) implies that $\Lambda(x) \neq 0, \forall x \in]-\varepsilon_0, +\varepsilon_0[^n, x \neq 0$ and property 5 (see Section 2) of degree ensures that $\varepsilon \to \deg(\Lambda(.),]-\varepsilon, +\varepsilon[^n, 0)$ is constant on $(0, +\varepsilon_0]$.

Let $r_0 > 0$ be such that $B_{r_0} \subset]-\varepsilon_0, +\varepsilon_0[$. Then $0 \notin \Lambda(\hat{\partial} B_r)$ and $\zeta(\Lambda, 0) = \deg(\Lambda(.), B_r, 0)$ for all $0 < r \leqslant r_0$. Let $r \in (0, r_0]$ be given, there exists $\varepsilon \equiv \varepsilon(r) \in (0, r)$ such that $]-\varepsilon, +\varepsilon[^n \subset B_r$. It is also clear from (33) that $A_\Lambda(B_r) = \{0\} \subset]-\varepsilon, +\varepsilon[^n$. Then using property 5 (see Section 2) of degree, we obtain

$$\zeta(\Lambda, 0) = \deg(\Lambda(.),]-\varepsilon, +\varepsilon[^n, 0) = \deg(\Lambda(.),]-\bar{\varepsilon}, +\bar{\varepsilon}[^n, 0) = \prod_{i=1}^{n} \sigma_i. \qquad \square$$

Proposition 3. *Suppose that there exists* $\varepsilon_0 > 0$ *such that*

$$\Lambda_i(x)\Lambda_i(y) > 0, \quad \forall x \in P_i(-\varepsilon),\ y \in P_i(+\varepsilon),\ \varepsilon \in (0, \varepsilon_0],\ (i \in \{1, \ldots, n\}). \tag{34}$$

Then

$$\zeta(\Lambda, 0) = 0.$$

Proof. Let $r_0 > 0$ such that $B_r \subset]-\varepsilon_0, +\varepsilon_0[^n$ $(0 < r \leqslant r_0)$. As in the proof of Proposition 2 we see that $\zeta(\Lambda, 0) = \deg(\Lambda(.), B_r, 0)$ for all $0 < r \leqslant r_0$. Let $r \in (0, r_0)$ be given. There exists $\varepsilon \equiv \varepsilon(r) \in (0, r)$ such that

$$\deg(\Lambda(.), B_r, 0) = \deg(\Lambda(.),]-\varepsilon, +\varepsilon[^n, 0).$$

Let us now consider the continuous homotopy $H : [0, 1] \times [-\varepsilon, +\varepsilon]^n \to \mathbb{R}^n$ defined by

$$H_i(\lambda, x) = \lambda\left(\Lambda_i(-\varepsilon e_i) + \frac{x_i + \varepsilon}{2\varepsilon}(\Lambda_i(\varepsilon e_i) - \Lambda_i(-\varepsilon e_i))\right) + (1 - \lambda)\Lambda_i(x),$$

where e_i denotes the ith unit vector of $\mathbb{R}^n$. We claim that if $x \in \partial(]-\varepsilon, +\varepsilon[^n)$ and $\lambda \in [0, 1]$ then $H(\lambda, x) \neq 0$. Indeed, if $x \in \partial(]-\varepsilon, +\varepsilon[^n)$ then either there exists $k \in \{1, \ldots, n\}$ such that $x \in P_k(-\varepsilon)$ or there exists $l \in \{1, \ldots, n\}$ such that $x \in P_l(+\varepsilon)$. If $x \in P_k(-\varepsilon)$ then $x_k = -\varepsilon$ and $H_k(\lambda, x) = \lambda\Lambda_k(-\varepsilon e_k) + (1 - \lambda)\Lambda_k(x)$. Here x and $-\varepsilon e_k \in P_k(-\varepsilon)$ and the continuity of Λ together with (34) ensure that $\Lambda_k(-\varepsilon e_k)$ and $\Lambda_k(x)$ are nonzero and have the same sign. Thus $H_k(\lambda, x) \neq 0$ and therefore $H(\lambda, x) \neq 0$. If $x \in P_l(+\varepsilon)$ then $x_l = +\varepsilon$ and $H_l(\lambda, x) = \lambda\Lambda_l(\varepsilon e_l) + (1 - \lambda)\Lambda_l(x)$. Here x and $\varepsilon e_l \in P_l(-\varepsilon)$ and the continuity of Λ together with (34) ensure that $\Lambda_l(\varepsilon e_l)$ and $\Lambda_l(x)$ are nonzero and have the same sign. It results that $H_l(\lambda, x) \neq 0$ and consequently $H(\lambda, x) \neq 0$. Thus from property 2 in Section 2, we obtain

$$\begin{aligned}\zeta(\Lambda, 0) &= \deg(\Lambda(.),]-\varepsilon, +\varepsilon[^n, 0)\\ &= \deg(H(0, .),]-\varepsilon, +\varepsilon[^n, 0) = \deg(H(1, .),]-\varepsilon, +\varepsilon[^n, 0).\end{aligned}$$

It is now clear that $H(1, .) \neq 0$ on $]-\varepsilon, +\varepsilon[^n$. Indeed,

$$\begin{aligned}H_i(1, x) &= \Lambda_i(-\varepsilon e_i) + \frac{x_i + \varepsilon}{2\varepsilon}(\Lambda_i(\varepsilon e_i) - \Lambda_i(-\varepsilon e_i))\\ &= \left(1 - \frac{x_i + \varepsilon}{2\varepsilon}\right)\Lambda_i(-\varepsilon e_i) + \frac{x_i + \varepsilon}{2\varepsilon}\Lambda_i(\varepsilon e_i),\end{aligned}$$

and $(x_i + \varepsilon)/2\varepsilon \in (0, 1)$ for $x \in]-\varepsilon, +\varepsilon[^n$. Thus $H_i(1, x) \neq 0$ since condition (34) entails that $\Lambda_i(-\varepsilon e_i)$ and $\Lambda_i(\varepsilon e_i)$ are nonzero and have same sign. It results from property 1 in Section 2, that $\deg(H(1, .),]-\varepsilon, +\varepsilon[^n, 0) = 0$. The conclusion follows. □

Example 2. (i) Let F and φ be defined by $F(x_1, x_2) = (2x_1, 3x_2)$ and $\varphi(x) = \Psi_{\mathbb{R}_+ \times \mathbb{R}_+}$. Here

$$\Lambda(x) = (x_1 - \max\{0, -x_1\}, x_2 - \max\{0, -2x_2\}).$$

For $\varepsilon > 0$, we see that $\Lambda_1(x) = -2\varepsilon < 0$ if $x \in P_1(-\varepsilon)$, $\Lambda_1(x) = \varepsilon > 0$ if $x \in P_1(+\varepsilon)$, $\Lambda_2(x) = -3\varepsilon < 0$ if $x \in P_2(-\varepsilon)$ and $\Lambda_2(x) = \varepsilon > 0$ if $x \in P_2(+\varepsilon)$. Applying Proposition 2, we get $\zeta(\Lambda, 0) = 1$.

(ii) Let F and φ be defined by $F(x_1, x_2) = (-2x_1 + x_2, -x_2)$ and $\varphi(x) = \Psi_{\mathbb{R}_+ \times \mathbb{R}_+}$. Here

$$\Lambda(x) = (x_1 - \max\{0, 3x_1 - x_2\}, x_2 - \max\{0, 2x_2\}).$$

For $\varepsilon > 0$, we see that $\Lambda_1(x) = -\varepsilon < 0$ if $x \in P_1(-\varepsilon)$, $\Lambda_1(x) = x_2 - 2\varepsilon \leqslant -\varepsilon < 0$ if $x \in P_1(+\varepsilon)$, $\Lambda_2(x) = -\varepsilon < 0$ if $x \in P_2(-\varepsilon)$ and $\Lambda_2(x) = -\varepsilon < 0$ if $x \in P_2(+\varepsilon)$. Applying Proposition 3, we obtain $\zeta(\Lambda, 0) = 0$.

The use of Propositions 2 and 3 is of particular interest if $\varphi \equiv \Psi_{\mathbb{R}^n_+}$ since in this case Λ_i can be easily evaluated by the formula

$$\Lambda_i(x) = x_i - \max\{0, x_i - F_i(x)\} = \min\{x_i, F_i(x)\}. \tag{35}$$

Let us now prove our main result by following a methodology that has been originated by Quittner [24] in the framework of parabolic variational inequalities involving unilateral constraints defined by a closed and convex set.

Theorem 5. *Suppose that assumptions* (h_1)–(h_7) *are satisfied. If the trivial stationary solution of* (19) *is asymptotically stable then*

$$\zeta(\Lambda, 0) = 1.$$

Proof. The mapping $F(.)$ is locally Lipschitz at 0 and thus there exists $\delta_0 > 0$ and $C_0 > 0$ such that

$$\|F(x) - F(y)\| \leqslant C_0\|x - y\|, \quad \forall x, y \in B_{\delta_0}. \tag{36}$$

From assumption (h_4), we see that there exists $\delta_1 > 0$ and $C_1 > 0$ such that

$$|\varphi(x) - \varphi(y)| \leqslant C_1\|x - y\|, \quad \forall x, y \in \mathrm{dom}\{\varphi\} \cap B_{\delta_1}. \tag{37}$$

The trivial stationary solution is isolated and thus, there exists $\delta_2 > 0$ such that

$$x \neq P_\varphi(x - F(x)), \quad \forall x \in B_{\delta_2}\backslash\{0\}. \tag{38}$$

The trivial stationary solution is stable and thus, there exists $\delta_3 > 0$ such that

$$|S(t)x_0| \leqslant \min\left\{1, \delta_0, \frac{\delta_1}{2(C_0 + 1)}\right\}, \quad \forall x_0 \in B_{\delta_3} \cap D(\partial\varphi). \tag{39}$$

The trivial stationary solution is attractive and thus, there exists $\delta_4 > 0$ such that

$$\lim_{t\to+\infty} S(t)x_0 = 0, \quad \forall x_0 \in B_{\delta_4} \cap D(\partial\varphi). \tag{40}$$

Set

$$r_0 := \min\left\{\frac{\delta_0}{2}, \frac{\delta_1}{2(1 + C_0)}, \delta_2, \delta_3, \delta_4\right\}.$$

Let $0 < r \leqslant r_0$ be given. From (38), we get the existence of $\eta := \eta(r) > 0$ such that

$$\|x - P_\varphi(x - F(x))\| \geqslant \eta, \quad \frac{r}{8} \leqslant \|x\| \leqslant r. \tag{41}$$

We set

$$G := \mathrm{id}_{\mathbb{R}^n} - F, \tag{42}$$

so that $\Lambda = \mathrm{id}_{\mathbb{R}^n} - P_\varphi(G)$.

To prove our result, we will use the following claims. The proofs of these claims are given in the appendix.

Claim 1.

$$\deg(\mathrm{id}_{\mathbb{R}^n} - S(t)P_{D(\partial\varphi)}., B_r, 0) = 1, \quad \forall t > 0,$$

where $S(.)$ *is defined in Section* 5.

Claim 2.

$$\deg(\mathrm{id}_{\mathbb{R}^n} - P_\varphi G(P_{D(\partial\varphi)}.), B_r, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - (\mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1}(\mathrm{id}_{\mathbb{R}^n} + tG)P_{D(\partial\varphi)}., B_r, 0), \quad \forall t > 0.$$

Claim 3.

$$\deg(\mathrm{id}_{\mathbb{R}^n} - P_\varphi G(P_{D(\partial\varphi)}.), B_r, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - P_\varphi G(.), B_r, 0).$$

Claim 4. *Let $x \in \hat{\partial} B_r (r \leqslant r_0)$ be given. We set*

$$u(t) := S(t)P_{D(\partial\varphi)}x, \quad t \geqslant 0$$

and

$$v(t) := (\mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1}(\mathrm{id}_{\mathbb{R}^n} + tG)P_{D(\partial\varphi)}x, \quad t > 0.$$

The following estimations hold:

(a) $\|P_{D(\partial\varphi)}x\| \leqslant r_0$;
(b) $\|P_\varphi G(P_{D(\partial\varphi)}x)\| \leqslant \delta_1$;
(c) $\|u(t)\| \leqslant \min\{1, \delta_0, \delta_1/2(C_0+1)\}, \forall t \geqslant 0$;
(d) $\|u(t) - P_{D(\partial\varphi)}x\| \leqslant \theta\sqrt{t}, \forall t \geqslant 0$, *with*

$$\theta := \sqrt{2(r_0+1)(C_1 + \bar{\omega}(r_0+1) + C_0 r_0 + \|F(0)\|)};$$

(e) $\|P_\varphi G(u(t))\| \leqslant \delta_1, \forall t \geqslant 0$;
(f) *Let $t^* \in (0, (1/2\bar{\omega})\ \ln(4)]$ be given. If $\|u(t^*)\| \geqslant r/4$ then $\|u(t)\| \geqslant r/8, \forall t \in (0, t^*]$.*
(g) $v(t) \in D(\hat{\partial}\varphi)$ *and*

$$\left\langle \frac{v(t) - P_{D(\partial\varphi)}x}{t}, w - v(t) \right\rangle + \langle v(t) - G(P_{D(\partial\varphi)}x), w - v(t)\rangle + \varphi(w) - \varphi(v(t)) \geqslant 0, \quad \forall w \in \mathbb{R}^n, \ \forall t > 0;$$

(h) $\|v(t) - P_{D(\partial\varphi)}x\| \leqslant \overline{\theta}\sqrt{t}; \forall t > 0$, *with*

$$\overline{\theta} := \sqrt{\frac{(C_0 r_0)^2}{2} + \|F(0)\| r_0 + C_1 r_0}.$$

Set

$$T^* := \min\left\{ \left(\frac{\delta_0}{2\overline{\theta}}\right)^2, \left(\frac{\delta_1}{2(1+C_0)\overline{\theta}}\right)^2 \right\}.$$

Then
(i) $\|v(t)\| \leqslant \min\{\delta_0, \delta_1/(1+C_0)\}, \forall t \in (0, T^*]$;
(j) $\|P_\varphi(G(v(t))\| \leqslant \delta_1, \forall t \in (0, T^*]$.

Claim 5. *Let* $0 < r \leqslant r_0$ *be given and set*

$$C := \max\{\tfrac{2}{3}\theta(1+C_0)(\|F(0)\|+C_0+C_1), \overline{\theta}(1+C_0)(\|F(0)\| + \delta_1 + (1+C_0)\delta_0 + C_1)\}$$

and

$$\overline{T} := \min\left\{\frac{1}{2\bar{\omega}}\ln(4), T^*, \frac{\eta^4}{2C^2(v^*+1)^2}, \frac{\eta^4}{2C^2(\frac{1}{v_*}+1)^2}\right\},$$

with $v^* = \frac{4}{3}(\delta_0 + r)/r$, $v_* = 7r/8(1+r)$, $\bar{\omega}$ *in Theorems* 1 *and* 4, T^* *in Claim* 4(h) *and* η *in* (41). *For any* $t \in (0, \overline{T}]$, *we have*

$$\deg(\mathrm{id}_{\mathbb{R}^n} - (\mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1}(\mathrm{id}_{\mathbb{R}^n} + tG)P_{D(\partial\varphi)}., B_r, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - S(t)P_{D(\partial\varphi)}., B_r, 0).$$

We are now able to use Claims 1–5 in order to conclude the proof of our result.

Let $0 < r \leqslant r_0$ be given and compute $\overline{T} \equiv \overline{T}(r)$ as in Claim 5. We have

$$\zeta(\Lambda, 0) = \deg(\Lambda, B_r, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - P_\varphi G(.), B_r, 0). \tag{43}$$

From Claim 3 and (43), we see that

$$\zeta(\Lambda, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - P_\varphi G(P_{D(\partial\varphi)}.), B_r, 0). \tag{44}$$

Let us now choose $t \in (0, \overline{T}]$. From Claim 2 and (44), we obtain

$$\zeta(\Lambda, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - (\mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1}(\mathrm{id}_{\mathbb{R}^n} + tG)P_{D(\partial\varphi)}., B_r, 0). \tag{45}$$

Claim 5 and (45) ensure that

$$\zeta(\Lambda, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - S(t)P_{D(\partial\varphi)}., B_r, 0). \tag{46}$$

Finally, Claim 1 and (46) yield

$$\zeta(\Lambda, 0) = 1. \qquad \square \tag{47}$$

The following result is an equivalent formulation of Theorem 5.

Corollary 1. *Suppose that assumptions* (h_1)–(h_7) *are satisfied. If* $\zeta(\Lambda, 0) \neq 1$ *then the trivial stationary solution of* (19) *is not asymptotically stable.*

Example 3. (i) The necessary condition of asymptotic stability $\zeta(\Lambda, 0) = 1$ is satisfied with the data given in Examples 1(i) and 2(i).

(ii) The trivial stationary solution of (19) with the data given in Example 1(ii) or Example 2(ii) is not asymptotically stable.

7. The case $\varphi = \psi_K$ with K closed and convex

It is known that if K is a polyhedron, then the system in (19) is equivalent to a complementarity system [12], whose study has a significant interest for control applications. Let us first state a result of Quittner [24].

Lemma 2 (*Quittner*). *Let $K \subset \mathbb{R}^n$ be a closed and convex set such that $0 \in K$. Set*

$$K_0 := \overline{\bigcup_{\alpha>0} \alpha K}. \tag{48}$$

Suppose that $G : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

(1) *G is continuous*;
(2) $G(0) = 0$;
(3) *$J_G(0)$ exists*;
(4) $\deg(\mathrm{id}_{\mathbb{R}^n} - P_{K_0}(J_G(0).), B_r, 0)$ *exists for $r > 0$ small enough.*

Then, for all $r > 0$ sufficiently small, $0 \notin (\mathrm{id}_{\mathbb{R}^n} - P_K(G(.)))(\partial B_r)$ and

$$\deg(\mathrm{id}_{\mathbb{R}^n} - P_K(G(.)), B_r, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - P_{K_0}(J_G(0).), B_r, 0).$$

Note that the original result of Quittner has been stated in a more general framework for compact continuous mapping G defined on Hilbert spaces.

Let us now remark that Lemma 2 can be used together with Corollary 1 and Propositions 2 and 3.

Example 4. Let F and K be defined by $F(x_1, x_2) = (-2\sin(x_1) + x_2, -\sin(x_2))$ and $K = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leqslant x_1,\ 0 \leqslant x_2 \leqslant 1 - x_1\}$. Then

$$K_0 = \mathbb{R}_+ \times \mathbb{R}_+$$

and

$$J_F(0) = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}.$$

Using the results of Example 2(ii) and Lemma 2, we get $\zeta(\Lambda, 0) = 0$. It results that the trivial stationary solution of (19) is not asymptotically stable.

Let us here remark that if $K \subset \mathbb{R}^n$ is a closed convex set with $0 \in K$ then $\varphi := \psi_K$ satisfies conditions (h_1)–(h_4) of Section 6. We see also that if the assumptions of Lemma 2 are satisfied for $G := \mathrm{id}_{\mathbb{R}^n} - F$ then condition (h_7) is also satisfied.

Theorem 6. *Let $K \subset \mathbb{R}^n$ be a closed and convex set such that $0 \in K$ and $K_0 \backslash \{0\} \neq \emptyset$ with K_0 in* (48). *Suppose that conditions* (h_5)–(h_6) *are satisfied. Suppose also that $F(0) = 0$ and $J_F(0)$ exists.*

If there exists $\mu < 0$ *and* $U_\mu \in K_0\backslash\{0\}$ *such that*

(1) $\langle U_\mu, h\rangle > 0, \forall h \in K_0\backslash\{0\}$;
(2) $J_F(0)^{\mathrm{T}} U_\mu = \mu U_\mu$.

Then the trivial stationary solution of (19) *is not asymptotically stable.*

Proof. Set $A := I - J_F(0)$ and define $h : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ by $h(\lambda, x) = \lambda x - P_{K_0}(\lambda Ax + (1-\lambda)U_\mu)$. We claim that if $x \in \partial B_r, r > 0$ and $\lambda \in [0, 1]$ then $h(\lambda, x) \neq 0$. Indeed, suppose on the contrary that

$$\lambda x = P_{K_0}(\lambda Ax + (1 - \lambda)U_\mu).$$

Then $\lambda \neq 0$ since otherwise get $0 = P_{K_0}(U_\mu) = U_\mu$ and a contradiction. Thus $\lambda \neq 0$ and since K_0 is a cone, we get

$$x = \frac{1}{\lambda} P_{K_0}(\lambda Ax + (1 - \lambda)U_\mu) = P_{K_0}\left(Ax + \frac{(1-\lambda)}{\lambda} U_\mu\right).$$

Thus $x \in K_0$ and

$$\left\langle x - Ax - \frac{(1-\lambda)}{\lambda} U_\mu, v - x \right\rangle \geqslant 0, \quad \forall v \in K_0.$$

Setting $v := x + U_\mu \in K_0 + K_0 \subset K_0$, we obtain

$$\langle J_F(0)x, U_\mu\rangle \geqslant \frac{(1-\lambda)}{\lambda} \|U_\mu\|^2 \geqslant 0.$$

Thus

$$\langle x, J_F(0)^{\mathrm{T}} U_\mu\rangle = \mu\langle x, U_\mu\rangle \geqslant 0.$$

Here since $x \in \partial B_r, r > 0$, then $x \in K_0\backslash\{0\}$ and thus from assumption (1), we obtain that $\langle x, U_\mu\rangle > 0$ so that $\mu \geqslant 0$ and a contradiction.

Then

$$\begin{aligned}\deg(\mathrm{id}_{\mathbb{R}^n} - P_{K_0}(\mathrm{id}_{\mathbb{R}^n} - J_F(0).), B_r, 0) &= \deg(h(1, .), B_r, 0)\\ &= \deg(h(0, .), B_r, 0) = \deg(-P_{K_0}(U_\mu), B_r, 0) = \deg(-U_\mu, B_r, 0) = 0.\end{aligned}$$

Using Lemma 2, we see that $\zeta(\Lambda, 0) = 0$ and the conclusion follows from Corollary 1. □

Example 5. Let F and K be defined by $F(x_1, x_2) = (-2\sin(x_1) - 10x_2, \cos(x_1) - \sin(x_2) - 1)$ and $K = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leqslant x_1, 0 \leqslant x_2 \leqslant 1 - x_1\}$. Then

$$K_0 = \mathbb{R}_+ \times \mathbb{R}_+$$

and

$$J_F(0) = \begin{pmatrix} -2 & -10 \\ 0 & -1 \end{pmatrix}.$$

Theorem 6 can be applied to ensure that the trivial stationary solution of (19) is not asymptotically stable. It is indeed easy to see that $\mu = -2$ is an eigenvalue of $J_F(0)^{\mathrm{T}}$ and that the corresponding eigenvector $U_\mu = (1, 10)^{\mathrm{T}}$ satisfies conditions (1) and (2) of Theorem 6.

Theorem 7. *Let $K \subset \mathbb{R}^n$ be a closed and convex set such that $0 \in K$ and $K_0^* \cap K_0 \backslash \{0\} \neq \emptyset$. Suppose that conditions* (h_5)–(h_6) *are satisfied. Suppose also that $F(0) = 0$, $J_F(0)$ exists and*

(1) $\langle J_F(0)x, x\rangle < 0, \forall x \in K_0, x \neq 0,$

then the trivial stationary solution of (19) *is not asymptotically stable.*

Proof. Let $U \in K_0^* \cap K_0 \backslash \{0\}$ be fixed. Set $A := I - J_F(0)$ and define $h : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ by $h(\lambda, x) = \lambda x - P_{K_0}(\lambda Ax + (1 - \lambda)U)$. We claim that if $x \in \partial B_r$, $r > 0$ and $\lambda \in [0, 1]$ then $h(\lambda, x) \neq 0$. Indeed, suppose on the contrary that $\lambda x = P_{K_0}(\lambda Ax + (1 - \lambda)U)$. It is clear that $\lambda \neq 0$. Indeed, suppose that $\lambda = 0$. Then $0 = P_{K_0}U = U$ and a contradiction. Thus $\lambda \neq 0$ and

$$x = P_{K_0}\left(Ax + \frac{(1-\lambda)}{\lambda}U\right).$$

Thus $x \in K_0$ and

$$\left\langle x - Ax - \frac{(1-\lambda)}{\lambda}U, v - x\right\rangle \geqslant 0, \quad \forall v \in K_0.$$

Setting $v := 2x$, we obtain since $U \in K_0^*$

$$\langle J_F(0)x, x\rangle \geqslant \frac{(1-\lambda)}{\lambda}\langle U, x\rangle \geqslant 0. \tag{49}$$

Here $x \in K_0 \backslash \{0\}$ and from assumption (1), we obtain $\langle J_F(0)x, x\rangle < 0$ which is a contradiction to (49).

Then

$$\begin{aligned}\deg(\mathrm{id}_{\mathbb{R}^n} - P_{K_0}(\mathrm{id}_{\mathbb{R}^n} - J_F(0).), B_r, 0) &= \deg(h(1, .), B_r, 0)\\ = \deg(h(0, .), B_r, 0) &= \deg(-P_{K_0}(U), B_r, 0) = \deg(-U, B_r, 0) = 0.\end{aligned}$$

Using Lemma 2, we see that $\zeta(\Lambda, 0) = 0$ and the conclusion follows from Corollary 1. □

Remark 5. It is clear that if the matrix $J_F(0)$ is negative definite, then assumption (1) in Theorem 7 is satisfied.

Example 6. Let F and K be defined as in Example 4. Theorem 7 can also be applied to ensure that the trivial stationary solution of (19) is not asymptotically stable. Indeed, here $K_0 \cap K_0^* \backslash \{0\} = \mathbb{R}_+ \times \mathbb{R}_+ \backslash \{(0, 0)\}$ and the matrix $J_F(0)$ is negative definite.

8. The case φ convex and dom$\{\varphi\} = \mathbb{R}^n$

Let us here suppose that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex with dom$\{\varphi\} = \mathbb{R}^n$. Then φ is continuous and $D(\partial\varphi) = \mathbb{R}^n$ (see Theorem 10.1 in [26]). Moreover φ is Lipschitzian, relative to every bounded set (see Theorem 10.4 in [26]). It results that φ satisfies conditions (h_1)–(h_4) of Section 6.

Proposition 4. *Let $L > 0$ be given and assume that $G : [0, L] \times \mathbb{R}^n \to \mathbb{R}^n; (\lambda, y) \mapsto G(\lambda, y)$ is a continuous function on $[0, L] \times \mathbb{R}^n$. Then the application*

$$(\lambda, y) \mapsto P_{\lambda\varphi}(G(\lambda, y))$$

is continuous on $[0, L] \times \mathbb{R}^n$.

Proof. Let $\{y_n\} \subset \mathbb{R}^n$ and $\{\lambda_n\} \subset [0, L]$ be given sequences such that $y_n \to y^*$ and $\lambda_n \to \lambda^*$ as $n \to +\infty$. We claim that $P_{\lambda_n\varphi}(G(\lambda_n, y_n)) \to P_{\lambda^*\varphi}(G(\lambda^*, y^*))$ as $n \to +\infty$. Indeed, setting $x_n := P_{\lambda_n\varphi}(G(\lambda_n, y_n))$ and $x^* := P_{\lambda^*\varphi}(G(\lambda^*, y^*))$, we have

$$\langle x_n - G(\lambda_n, y_n), v - x_n\rangle + \lambda_n\varphi(v) - \lambda_n\varphi(x_n) \geqslant 0, \quad \forall v \in \mathbb{R}^n \tag{50}$$

and

$$\langle x^* - G(\lambda^*, y^*), v - x^*\rangle + \lambda^*\varphi(v) - \lambda^*\varphi(x^*) \geqslant 0, \quad \forall v \in \mathbb{R}^n. \tag{51}$$

Let us first check that the sequence $\{x_n\}$ is bounded. Indeed, suppose on the contrary that $\|x_n\| \to +\infty$ as $n \to +\infty$. Setting $v := 0$ in (50), we obtain

$$-\langle x_n - G(\lambda_n, y_n), x_n\rangle + \lambda_n[\varphi(0) - \varphi(x_n)] \geqslant 0$$

and thus

$$\|x_n\|^2 \leqslant \|G(\lambda_n, y_n)\|\|x_n\| + \lambda_n[\varphi(0) - \varphi(x_n)].$$

It results that

$$1 \leqslant \frac{\|G(\lambda_n, y_n)\|}{\|x_n\|} + \frac{\lambda_n}{\|x_n\|^2}[\varphi(0) - \varphi(x_n)].$$

For n large enough, $1/\|x_n\| \in (0, 1]$ and using the convexity of φ, we get

$$\varphi\left(\frac{x_n}{\|x_n\|}\right) \leqslant \frac{1}{\|x_n\|}\varphi(x_n) + \left(1 - \frac{1}{\|x_n\|}\right)\varphi(0)$$

and thus

$$\frac{\varphi(0) - \varphi(x_n)}{\|x_n\|} \leqslant \varphi(0) - \varphi\left(\frac{x_n}{\|x_n\|}\right).$$

We obtain

$$1 \leqslant \frac{\|G(\lambda_n, y_n)\|}{\|x_n\|} + \lambda_n\left[\frac{\varphi(0) - \varphi(x_n/\|x_n\|)}{\|x_n\|}\right]. \tag{52}$$

Here

$$\lim_{n\to+\infty} \frac{\varphi(x_n/\|x_n\|)}{\|x_n\|} = 0$$

since the sequence $x_n/\|x_n\|$ remains in a compact set and φ is continuous. Taking now the limit as $n \to +\infty$ in (52), we obtain the contradiction $1 \leqslant 0$. The sequence $\{x_n\}$ is thus bounded.

Setting $v := x^*$ in (50) and $v := x_n$ in (51), we obtain the relations

$$\langle x_n - G(\lambda_n, y_n), x_n - x^*\rangle - \lambda_n\varphi(x^*) + \lambda_n\varphi(x_n) \leqslant 0 \tag{53}$$

and

$$-\langle x^* - G(\lambda^*, y^*), x_n - x^*\rangle - \lambda^*\varphi(x_n) + \lambda^*\varphi(x^*) \leqslant 0. \tag{54}$$

Thus

$$\begin{aligned}\|x_n - x^*\|^2 &\leqslant \|G(\lambda_n, y_n) - G(\lambda^*, y^*)\|\|x_n - x^*\| \\ &\quad + (\lambda_n - \lambda^*)\varphi(x^*) + (\lambda^* - \lambda_n)\varphi(x_n).\end{aligned} \tag{55}$$

The sequence $\{x_n\}$ is bounded and thus the sequence $\{\varphi(x_n)\}$ is bounded too since φ is continuous. Moreover $\|G(\lambda_n, y_n) - G(\lambda^*, y^*)\| \to 0$ and $(\lambda_n - \lambda^*) \to 0$ as $n \to +\infty$. Then (55) yields $x_n \to x^*$ as $n \to +\infty$. □

Proposition 5. *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Suppose also that there exists $\sigma > 0$ and a continuous mapping $H : \mathbb{R}^n \to \mathbb{R}^n$ such that*

(1) $\Lambda(x) \neq 0, \forall x \in \bar{B}_\sigma, x \neq 0$;
(2) $H(x) \neq 0, \forall x \in \bar{B}_\sigma, x \neq 0$;
(3) $\langle F(x), H(x)\rangle + \varphi(x) - \varphi(x - H(x)) \geqslant 0, \forall x \in \bar{B}_\sigma$.

Then

$$\zeta(\Lambda, 0) = \deg(H, B_r, 0), \quad \forall r \in (0, \sigma].$$

Proof. Let $r \in (0, \sigma]$ be given. Let $h : [0, 1] \times \bar{B}_r \to \mathbb{R}^n; (\lambda, y) \to h(\lambda, y) := y - P_{\lambda\varphi}(y - \lambda F(y) - (1 - \lambda)H(y))$. Proposition 4 ensures that h is continuous. Let us now check that $h(\lambda, x) \neq 0, \forall x \in \partial B_r, \lambda \in [0, 1]$. Indeed, suppose on the contrary that there exists $x \in \mathbb{R}^n$, $\|x\| = r$ and $\lambda \in [0, 1]$ such that $h(\lambda, x) = 0$, that is

$$x = P_{\lambda\varphi}(x - \lambda F(x) - (1 - \lambda)H(x)).$$

We first remark that $\lambda \neq 0$. Indeed, suppose that $\lambda = 0$. Then $x = P_0(x - H(x)) = x - H(x)$. This yields $H(x) = 0$ which is a contradiction to assumption (2) since here $x \neq 0$. We remark now that $\lambda \neq 1$. Indeed, suppose that $\lambda = 1$. Then $x = P_\varphi(x - F(x))$ and thus $\Lambda(x) = 0$ which is a contradiction to assumption (1) since here $x \neq 0$.

Thus $0 < \lambda < 1$ and

$$\langle \lambda F(x) + (1 - \lambda)H(x), v - x\rangle + \lambda\varphi(v) - \lambda\varphi(x) \geqslant 0, \quad \forall v \in \mathbb{R}^n.$$

Setting $v := x - H(x)$, we obtain

$$\lambda[\langle F(x), H(x)\rangle + \varphi(x) - \varphi(x - H(x))] \leqslant -(1-\lambda)\|H(x)\|^2 < 0,$$

which is a contradiction to assumption (3).

Thus

$$\begin{aligned}\deg(\mathrm{id}_{\mathbb{R}^n} - P_\varphi(\mathrm{id}_{\mathbb{R}^n} - F), B_r, 0) &= \deg(h(1,.), B_r, 0)\\ &= \deg(h(0,.), B_r, 0)\\ &= \deg(\mathrm{id}_{\mathbb{R}^n} - P_0(\mathrm{id}_{\mathbb{R}^n} - H), B_r, 0)\\ &= \deg(H, B_r, 0). \qquad \square\end{aligned}$$

Proposition 6. *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Suppose also that there exists $\sigma > 0$ and a continuous mapping $H : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\langle F(x), H(x)\rangle + \varphi(x) - \varphi(x - H(x)) > 0, \quad \forall x \in \bar{B}_\sigma, x \neq 0. \tag{56}$$

Then

$$\zeta(\Lambda, 0) = \deg(H, B_r, 0), \quad \forall r \in (0, \sigma].$$

Proof. The result is a consequence of Proposition 5. It is clear that if (56) holds then conditions (2) and (3) of Proposition 5 are satisfied. We claim that condition (1) of Proposition 5 holds too. Indeed, suppose on the contrary that there exists $x \in \bar{B}_\sigma \backslash \{0\}$ such that $\Lambda(x) = 0$. Then

$$\langle F(x), v - x\rangle + \varphi(v) - \varphi(x) \geqslant 0, \quad \forall v \in \mathbb{R}^n.$$

Setting $v = x - H(x)$, we get

$$\langle F(x), H(x)\rangle + \varphi(x) - \varphi(x - H(x)) \leqslant 0.$$

which is a contradiction to (56). $\square$

9. The case of linear complementarity problems

In view of applications like stabilization, we examine in this section the case when the vector field is linear invariant with matrix $A \in \mathbb{R}^{n\times n}$, and $\varphi(\cdot) = \Psi_K(\cdot)$ with $K = \mathbb{R}^n_+$. The following holds:

Proposition 7. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $F(x) = Ax$ with $A \in \mathbb{R}^{n\times n}$ and $\varphi(\cdot) = \Psi_K(\cdot)$ with $K = \mathbb{R}^n_+$. If A satisfies the conditions* (i) $A_{ii} > 0$, $1 \leqslant i \leqslant n$, *and* (ii) $|A_{ij}/A_{ii}| < 1/n$ *for all* $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $i \neq j$, *then* $\zeta(\lambda, 0) = 1$.

Proof. The proof uses Proposition 2. As we have seen in (35), we have $\Lambda_i(x) = \min\{x_i, e_i^{\mathrm{T}} Ax\}$, with e_i the ith unit vector of $\mathbb{R}^n$. Let us now calculate $\Lambda_i(x)$ and $\Lambda_i(y)$ for the

vectors x and y as in (33). Taking into account the definitions of the sets $P_i(-\varepsilon)$ and $P_i(+\varepsilon)$, $\varepsilon > 0$, we obtain

$$\Lambda_i(x) = \min\{-\varepsilon, e_i^{\mathrm{T}} Ax + A_{ii}(-\varepsilon - x_i)\}$$

with $x \in [-\varepsilon, +\varepsilon]^n$, and

$$\Lambda_i(y) = \min\{\varepsilon, e_i^{\mathrm{T}} Ay + A_{ii}(\varepsilon - y_i)\}$$

with $y \in [-\varepsilon, +\varepsilon]^n$ (in both these expressions we do not consider the variables x and y in the sets $P_i(-\varepsilon)$ and $P_i(+\varepsilon)$, respectively, but as free vectors of $\mathbb{R}^n$ satisfying $\|x\|_\infty (:= \max_{1 \leqslant i \leqslant n} \{|x_i|\}) \leqslant \varepsilon$ and $\|y\|_\infty \leqslant \varepsilon$). The condition $\Lambda_i(x)\Lambda_i(y) < 0$ can then be written as

$$\max\{\varepsilon, -e_i^{\mathrm{T}} Ax + A_{ii}x_i + A_{ii}\varepsilon\} \times \min\{\varepsilon, e_i^{\mathrm{T}} Ay - A_{ii}y_i + A_{ii}\varepsilon\} > 0. \tag{57}$$

The inequality in (57) is equivalent to

$$\text{(a)}\ \max\{\varepsilon, -e_i^{\mathrm{T}} Ax + A_{ii}x_i + A_{ii}\varepsilon\} > 0 \text{ and (b)}\ \min\{\varepsilon, e_i^{\mathrm{T}} Ay - A_{ii}y_i + A_{ii}\varepsilon\} > 0$$

or

$$\text{(c)}\ \max\{\varepsilon, -e_i^{\mathrm{T}} Ax + A_{ii}x_i + A_{ii}\varepsilon\} < 0 \text{ and (d)}\ \min\{\varepsilon, e_i^{\mathrm{T}} Ay - A_{ii}y_i + A_{ii}\varepsilon\} < 0$$

for all $\|x\|_\infty \leqslant \varepsilon$ and $\|y\|_\infty \leqslant \varepsilon$. The inequality in (c) is impossible to fulfill and the inequality in (a) is always satisfied, since $\varepsilon > 0$. We are therefore left with the inequality (b) only, which is satisfied if and only if $e_i^{\mathrm{T}} Ay - A_{ii}y_i > -A_{ii}\varepsilon$ for all $y \in \mathbb{R}^n$ with $\|y\|_\infty \leqslant \varepsilon$. Using assumption (i), we see that this condition can be rewritten as

$$\left(e_i^{\mathrm{T}} \frac{A}{A_{ii}} - e_i^{\mathrm{T}}\right) y + \varepsilon > 0 \tag{58}$$

for all $\|y\|_\infty \leqslant \varepsilon$. Equivalently

$$\left(e_i^{\mathrm{T}} \frac{A}{A_{ii}} - e_i^{\mathrm{T}}\right) y + 1 > 0 \tag{59}$$

for all $\|y\|_\infty \leqslant 1$. It is clear that inequality (59) is satisfied if and only if

$$\left|\left(e_i^{\mathrm{T}} \frac{A}{A_{ii}} - e_i^{\mathrm{T}}\right) y\right| < 1, \quad \forall y \in \mathbb{R}^n, \|y\|_\infty \leqslant 1. \tag{60}$$

Assumption (ii) ensures that

$$\left\| \frac{A^{\mathrm{T}}}{A_{ii}} e_i - e_i \right\|_\infty = \max_{1 \leqslant j \leqslant n} \left| \frac{A_{ij}}{A_{ii}} - \delta_{ij} \right| < \frac{1}{n},$$

where δ_{ij} is the Kronecker symbol. If $\|y\|_\infty \leqslant 1$ then

$$\left|\left(e_i^{\mathrm{T}} \frac{A}{A_{ii}} - e_i^{\mathrm{T}}\right) y\right| \leqslant n \left\| \frac{A^{\mathrm{T}}}{A_{ii}} e_i - e_i \right\|_\infty \|y\|_\infty \leqslant n \left\| \frac{A^{\mathrm{T}}}{A_{ii}} e_i - e_i \right\|_\infty < 1,$$

so that (60) is satisfied. Finally, Proposition 2 ensures that $\zeta(\Lambda, 0) = 1$. □

Remark 6. (i) From Proposition 3 in [12], it follows that if A is diagonal with positive entries, then the equilibrium $x=0$ is asymptotically stable. Incidentally one sees that $A_{ij}=0$, $i \neq j$, implies that $|A_{ij}/A_{ii}| = 0 < 1/n$ so that the necessary conditions for asymptotic stability are satisfied.

(ii) The conditions (i) and (ii) are not necessary conditions for asymptotic stability. Indeed matrices A which satisfy (i) and $A_{ij} \geqslant 0$ for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ are strictly copositive on K and guarantee asymptotic stability (see Proposition 3 in [12]).

(iii) Condition (ii) implies that A is diagonally dominant.

Example 7. Let us consider a planar case with $A \in \mathbb{R}^{2\times 2}$. The above conditions read $A_{11} > 0$, $A_{22} > 0$, $|A_{12}|/A_{11} < \frac{1}{2}$ and $|A_{21}|/A_{22} < \frac{1}{2}$. We see that A is a P-matrix [9] since both principal minors are positive, as $A_{ii} > 0$ $(i = 1, 2)$ and $\det(A) = A_{11}A_{22}(1 - A_{12}A_{21}/A_{11}A_{22}) > 0$. Notice that there exist P-matrices which are not positive definite, like for instance

$$A = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

[9, p. 147]. Also P-matrices may have complex eigenvalues with negative real parts, for instance

$$A = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}, \quad \lambda_1 = -\tfrac{1}{2} + \mathrm{j}\tfrac{\sqrt{3}}{2}, \quad \lambda_2 = -\tfrac{1}{2} - \mathrm{j}\tfrac{\sqrt{3}}{2}.$$

However, it can be shown that the above conditions guarantee that the eigenvalues of A have positive real parts. Let γ_1 and γ_2 be the two eigenvalues of A. Then $A_{11}A_{22} - A_{12}A_{21} = \gamma_1\gamma_2$ and $A_{22} + A_{11} = \gamma_1 + \gamma_2$. Since A is a P-matrix, its real eigenvalues are positive. Thus if the eigenvalues have negative real parts, they must be complex conjugate and in such a case $A_{22}A_{11} - A_{12}A_{21} = \gamma_1\gamma_2 > 0$ and $A_{22} + A_{11} = \gamma_1 + \gamma_2 < 0$ so that $A_{22} < -A_{11}$. This is a contradiction since $A_{22} > 0$ and $A_{11} > 0$. We conclude that in the planar case, the necessary conditions of Proposition 7 imply that A is a P-matrix with (positive) real eigenvalues.

We now make use of Proposition 3 to derive conditions for under which the stationary solution is not asymptotically stable.

Proposition 8. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $F(x) = Ax$ with $A \in \mathbb{R}^{n\times n}$ and $\varphi(\cdot) = \Psi_K(\cdot)$ with $K = \mathbb{R}^n_+$. If A satisfies conditions* (i) $A_{ii} < 0$, $1 \leqslant i \leqslant n$, *and* (ii) $|A_{ij}/A_{ii}| < 1/n$ *for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $i \neq j$, then $\zeta(\lambda, 0) = 0$ and the stationary solution of* (19) *is not asymptotically stable.*

Proof. The proof uses Proposition 3. The calculations are similar to those in the proof of Proposition 7. The conditions in (a)–(d) now are

(a′) $\max\{\varepsilon, -e_i^{\mathrm{T}}Ax + A_{ii}x_i + A_{ii}\varepsilon\} > 0$ and (b′) $\min\{\varepsilon, e_i^{\mathrm{T}}Ay - A_{ii}y_i + A_{ii}\varepsilon\} < 0$

or

(c′) $\max\{\varepsilon, -e_i^{\mathrm{T}}Ax + A_{ii}x_i + A_{ii}\varepsilon\} < 0$ and (d′) $\min\{\varepsilon, e_i^{\mathrm{T}}Ay - A_{ii}y_i + A_{ii}\varepsilon\} > 0$

for all $\|x\|_\infty \leqslant \varepsilon$ and $\|y\|_\infty \leqslant \varepsilon$. Inequality (c′) cannot be satisfied since $\varepsilon > 0$. Inequality (a′) is always satisfied. Inequality (b′) is equivalent to having $e_i^{\mathrm{T}} Ay - A_{ii} y_i + A_{ii}\varepsilon < 0$ for all $\|y\|_\infty \leqslant \varepsilon$. Using assumption (i), this can be rewritten as

$$\left(e_i^{\mathrm{T}} \frac{A}{A_{ii}} - e_i^{\mathrm{T}}\right) y + 1 > 0, \quad \forall y \in \mathbb{R}^n, \|y\|_\infty \leqslant 1. \tag{61}$$

As in the proof of Proposition 7, we see that assumption (ii) ensures that (61) holds. We conclude that both (i) and (ii) assure that $\Lambda_i(x)\Lambda_i(y) > 0$ for all $x \in P_i(-\varepsilon)$ and $y \in P_i(+\varepsilon)$ and all $1 \leqslant i \leqslant n$, so that $\zeta(\Lambda, 0) = 0$ and the stationary solution of (19) cannot be asymptotically stable. □

Application 1 (*Asymptotic stabilization*). Let us here consider a controlled system with transition matrix $A \in \mathbb{R}^{n\times n}$, input matrix $B \in \mathbb{R}^{n\times 1}$, and a feedback controller $u(.) = Ux(.)$ with $U \in \mathbb{R}^{1\times n}$. The corresponding mathematical model is of the form given in (19) with $F(x) = (A + BU)x$. Assume that the pair (A, B) has the so-called controller form [31], i.e. $A_{i,i+1} = 1$ and $A_{ij} = 0$ for all $j \neq i + 1$ and all $1 \leqslant i \leqslant n - 1$, $A_{nj} = \alpha_j$ where the α_j's are the coefficient of the characteristic polynomial of A, and $B = e_n$. One can show that the closed-loop system does not satisfy the conditions of Proposition 2. Or, that it does satisfy the conditions of Proposition 3. Consequently, a controlled evolution variational inequality as in (19) with A and B in a controller form cannot be asymptotically stabilized by constant feedback. This is a major discrepancy with unconstrained systems. This shows that having (A, B) a controllable pair (for the unconstrained system) is not at all sufficient to guarantee the asymptotic stabilization of a controlled variational inequality. This is in accordance with the results in [7] on controllability of planar variational inequalities.

Conditions (i) and (ii) of Proposition 7 now read as conditions for asymptotic stabilization: given a pair (A, B), find a matrix $U \in \mathbb{R}^{1\times n}$ such that

$$\begin{aligned} &\text{(a) } A_{ii} + B_i U_i > 0, \ 1 \leqslant i \leqslant n, \\ &\text{(b) } |A_{ij} + B_i U_j| < \frac{1}{n}(A_{ii} + B_i U_i), \ \forall i \neq j, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n, \end{aligned} \tag{62}$$

so that there exists a constant feedback control. We can now set some conditions to be satisfied by (A, B) so that the feedback guarantees $\zeta(\Lambda, 0) = 1$:

(1) $A \neq 0$ (from (62)(a) and combining with (62)(b)).
(2) If $A_{ii} = 0$ for some i, then necessarily $B_i \neq 0$ (from (62)(a)).
(3) If $B_i = 0$ for some i, then necessarily $A_{ii} \neq 0$ (from (62)(a)).
(4) Controllability of (A, B) (the Kalman matrix is of rank n) is not sufficient.
(5) Controllability of (A, B) (the Kalman matrix is of rank n) is not necessary: there exist pairs (A, B) such that the Kalman matrix has rank $< n$, and which satisfy the conditions in (62) or the sufficient condition of Proposition 7.

Item (5) can be proved by choosing a planar example with $A_{11} > 0$, $A_{12}{=}0$, $A_{21} \neq 0$, $A_{22} \neq 0$, and $B_1{=}0$, $B_2 \neq 0$. We have $\mathrm{rank}(B \ AB){=}1$, even if $0 < |A_{21}{+}B_2U_1| < \frac{1}{2}(A_{22}{+}B_2U_2)$. Let us choose $U_2 = \mathrm{sgn}(B_2)(\alpha - A_{22})/|B_2|$, $\alpha > 0$, and $U_1 = \mathrm{sgn}(B_2)(\beta - A_{21})/|B_2|$, $\beta \geqslant 0$,

$\beta < \alpha/2$. Then the sufficient condition of Remark 6(i) is satisfied if $\beta = 0$, and conditions (i) are (ii) of Proposition 7 are satisfied if $\beta < \alpha/2$.

10. Instability results

Inasmuch as stability is usually a desired property, it is important to dispose of some mathematical results which can be used to recognize instability. Here we use Theorem 5 together with Theorem 3 in order to state conditions ensuring instability.

Theorem 8. *Suppose that assumptions* (h_1)–(h_7) *are satisfied. Suppose in addition that there exists* $\sigma > 0$ *and* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1) $\varphi(.) - \varphi(. - V'(.))$ *is lower semicontinuous on* $D(\partial\varphi) \cap \overline{B}_\sigma$;
(2) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0, \forall x \in D(\partial\varphi) \cap \overline{B}_\sigma$;
(3) $\mathcal{M}(F, \varphi, V, \overline{B}_\sigma) = \{0\}$,

where (see Section 4) $\mathcal{M}(F, \varphi, V, \overline{B}_\sigma)$ *denotes the largest invariant subset of* $\mathcal{E}(F, \varphi, V) \cap \overline{B}_\sigma = \{x \in D(\partial\varphi) \cap \overline{B}_\sigma : \langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) = 0\}$.

If $\zeta(\Lambda, 0) \neq 1$ *then the trivial stationary solution of* (19) *is unstable*.

Proof. Suppose on the contrary that the trivial stationary solution of (19) is stable. Then we may find $\delta > 0$ such that if $x_0 \in \overline{B}_\delta$ then $\gamma(x_0) \subset \overline{B}_\sigma$. We may apply Theorem 3 (with $\Psi := \bar{B}_\sigma$) to get $\lim_{t\to+\infty} d(S(t)x_0, \mathcal{M}(F, \varphi, V, \bar{B}_\sigma)) = 0$. Here $\mathcal{M}(F, \varphi, V, \overline{B}_\sigma) = \{0\}$ and thus the trivial stationary solution of (19) is attractive. It results that the trivial stationary solution of (19) is asymptotically stable and we obtain a contradiction to $\zeta(\Lambda, 0) \neq 1$. □

Corollary 2. *Suppose that assumptions* (h_1)–(h_7) *are satisfied. Suppose in addition that there exists* $\sigma > 0$ *and* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1) $\varphi(.) - \varphi(. - V'(.))$ *is lower semicontinuous on* $D(\partial\varphi) \cap \overline{B}_\sigma$;
(2) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) > 0, \ x \in D(\partial\varphi) \cap \overline{B}_\sigma, x \neq 0$.

If $\zeta(\Lambda, 0) \neq 1$ *then the trivial stationary solution of* (19) *is unstable*.

Proof. The result is a direct consequence of Theorem 8 since condition (2) ensures that $\mathcal{M}(F, \varphi, V, \overline{B}_\sigma) = \{0\}$. □

The following result is a direct consequence of the previous one.

Corollary 3. *Let* $K \subset \mathbb{R}^n$ *be a closed convex set such that* $0 \in K$. *Suppose that assumptions* (h_5)–(h_7) *are satisfied. Suppose in addition that there exists* $\sigma > 0$ *and* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1) $V'(x) \in -K_\infty, x \in K \cap \overline{B}_\sigma$;
(2) $\langle F(x), V'(x)\rangle > 0, \ x \in K \cap \overline{B}_\sigma, x \neq 0$;

If $\zeta(\Lambda, 0) \neq 1$ *then the trivial stationary solution of* (19) *is unstable*.

Example 8. The trivial stationary solution of (19) with the data given in Example 2(ii) is unstable. Indeed, we know that here $\zeta(\Lambda, 0) = 0$ and it is easy to see that the conditions of Corollary 3 hold with $V(.) := -\frac{1}{2}\|.\|^2$.

Theorem 9. *Suppose that* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is convex and* $\mathrm{dom}\{\varphi\} = \mathbb{R}^n$. *Suppose that assumptions* (h_5)–(h_7) *are satisfied. Suppose in addition that there exists* $\sigma > 0$ *and* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1) $V'(x) \neq 0, \forall x \in \overline{B}_\sigma, x \neq 0$;
(2) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0, \quad x \in \overline{B}_\sigma$;
(3) $\mathscr{M}(F, \varphi, V, \overline{B}_\sigma) = \{0\}$;
(4) $\deg(V', B_r, 0) \neq 1$ *for* $r > 0$ *small enough.*

Then the trivial stationary solution of (19) *is unstable.*

Proof. The result is a consequence of Theorem 8 and Proposition 5 with $H := V'$. □

Corollary 4. *Suppose that* $\varphi : \mathbb{R}^n \xrightarrow{h} \mathbb{R}$ *is convex and* $\mathrm{dom}\{\varphi\} = \mathbb{R}^n$. *Suppose that assumptions* (h_5)–(h_7) *are satisfied. Suppose in addition that there exists* $\sigma > 0$ *and a nonsingular and symmetric matrix* $A \in \mathbb{R}^{n\times n}$ *such that*

$$\langle F(x), Ax\rangle + \varphi(x) - \varphi(x - Ax) \geqslant 0, \quad x \in \overline{B}_\sigma.$$

If $\mathscr{M}(F, \varphi, \frac{1}{2}\langle A., .\rangle, \overline{B}_\sigma) = \{0\}$ *and* $\det A < 0$ *then the trivial stationary solution of* (19) *is unstable.*

Proof. The result is a consequence of Theorem 9 with $V(.) = \frac{1}{2}\langle A., .\rangle$. Here $\deg(V', B_r, 0) = \deg(A., B_r, 0) = \mathrm{sgn}(\det A) = -1, \forall r > 0$. □

The following result is a direct consequence of Corollary 4.

Corollary 5. *Suppose that* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is convex and* $\mathrm{dom}\{\varphi\} = \mathbb{R}^n$. *Suppose that assumptions* (h_5)–(h_7) *are satisfied. Suppose in addition that there exists* $\sigma > 0$ *and a nonsingular and symmetric matrix* $A \in \mathbb{R}^{n\times n}$ *such that*

$$\langle F(x), Ax\rangle + \varphi(x) - \varphi(x - Ax) > 0, \quad x \in \overline{B}_\sigma \backslash \{0\}.$$

If $\det A < 0$ *then the trivial stationary solution of* (19) *is unstable.*

Example 9. Let F and φ be defined by

$$F(x) = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$\varphi(x) = d|x_2|,$$

with $a < 0$, $b > 0$, $c \in \mathbb{R}$ and $d > 0$. Setting

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

we see that $\langle F(x), Ax\rangle + \varphi(x) - \varphi(x - Ax) = -ax_1^2 + bx_2^2 + d|x_2| > 0$, $\forall (x_1, x_2) \in \mathbb{R}^2\backslash\{(0,0)\}$. Here $\det A = -1$ and thus the trivial stationary solution of (19) is unstable.

Application 2 (*Second-order dynamical systems in mechanics*). Let us here deal with the following class of second-order dynamical systems:

Let $\Phi : \mathbb{R}^m \to \mathbb{R}$ be a convex function with $\mathrm{dom}(\Phi) = \mathbb{R}^m$. Let $M, C, K \in \mathbb{R}^{m\times m}$ be given matrices. We consider the problem: Find a function $t \mapsto q(t)$ $(t \geqslant 0)$ with $q \in C^1([0, +\infty); \mathbb{R}^m)$, and such that

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} \in L^\infty_{\mathrm{loc}}(0, +\infty; \mathbb{R}^m), \tag{63}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} \text{ is right-differentiable on } [0, +\infty), \tag{64}$$

$$M\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}(t) + C\frac{\mathrm{d}q}{\mathrm{d}t}(t) + Kq(t) \in -\partial\Phi\left(\frac{\mathrm{d}q}{\mathrm{d}t}(t)\right), \quad \text{a.e. } t \geqslant 0. \tag{65}$$

The model in (65) can be used in Mechanics to describe the motion of various systems having frictional contact. For such problems, m is the number of degrees of freedom of the system, M is the mass matrix of the system, C is the viscous damping matrix of the system and K is the stiffness matrix. The term $\partial\Phi(.)$ is used to model the unilaterality of the contact induced by friction forces.

Let us first assume that:

(H_1) M is symmetric and positive definite;
(H_2) K is symmetric and nonsingular;
(H_3) $x^{\mathrm{T}}Cx + \Phi(x) - \Phi(0) > 0$, $\forall x \in \mathbb{R}^m$, $x \neq 0$;
(H_4) $0 \in \partial\Phi(0)$.

Let

$$\sigma(M, K) := \{\lambda \in \mathbb{C} : \det(\lambda M + K) = 0\} = \{\lambda_1, \ldots, \lambda_m\}.$$

Using assumptions (H_1) and (H_2), we may assert that (see e.g. [2]):

$$\sigma(M, K) \subset \mathbb{R}\backslash\{0\},$$

and there exists a nonsingular matrix $R \in \mathbb{R}^{m\times m}$ such that

$$R^{\mathrm{T}}MR = I \text{ and } R^{\mathrm{T}}KR = K_0,$$

where I denotes the $m \times m$ identity matrix and K_0 is the diagonal matrix defined by $(K_0)_{ii} = \lambda_i$, $(i \in \{1, \ldots, m\})$.

Setting

$$q := Rz,$$

we may rewrite (65) as follows:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2}(t) + R^{\mathrm{T}} CR \frac{\mathrm{d}z}{\mathrm{d}t}(t) + K_0 z(t) \in -R^{\mathrm{T}} \partial \Phi \left(R \frac{\mathrm{d}z}{\mathrm{d}t}(t) \right), \quad \text{a.e } t \geqslant 0. \tag{66}$$

Let us here define the convex function $\chi : \mathbb{R}^m \to \mathbb{R}$ by the formula

$$\chi(w) = (\Phi \circ R)(w), \quad \forall w \in \mathbb{R}^m.$$

Then (see Theorem 23.9 in [26]),

$$\partial \chi(w) = R^{\mathrm{T}} \partial \Phi(Rw), \quad \forall w \in \mathbb{R}^m.$$

Let us now set

$$x_1 := z, \quad x_2 := \frac{\mathrm{d}z}{\mathrm{d}t}, \quad x = (x_1 \quad x_2)^{\mathrm{T}}, \quad \dot{x}_1 = \frac{\mathrm{d}x_1}{\mathrm{d}t}, \quad \dot{x}_2 = \frac{\mathrm{d}x_2}{\mathrm{d}t}. \tag{67}$$

It is clear that (66) is equivalent to the following first-order system:

$$\begin{cases} \dot{x}_1 - x_2 = 0, \\ \dot{x}_2 + R^{\mathrm{T}} CR x_2 + K_0 x_1 \in -\partial \chi(x_2). \end{cases} \tag{68}$$

It results that our problem can be written as in (19) with $n = 2m$ and where F and φ are defined by

$$F(x) = \begin{pmatrix} 0 & -I \\ K_0 & R^{\mathrm{T}} CR \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$\varphi(x) = \chi(x_2).$$

Let V be defined by

$$V(x) = \tfrac{1}{2} \langle Ax, x \rangle,$$

where

$$A = \begin{pmatrix} K_0 & 0 \\ 0 & I \end{pmatrix}.$$

We see that

$$V'(x) = Ax \neq 0, \quad \forall x \in \mathbb{R}^n, x \neq 0, \tag{69}$$

$$\deg(V', B_r, 0) = \operatorname{sgn}(\det A) = \operatorname{sgn}(\det K_0), \quad \forall r > 0 \tag{70}$$

and

$$\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) = x_2^{\mathrm{T}} R^{\mathrm{T}} CR x_2 + \chi(x_2) - \chi(0) \geqslant 0, \quad \forall x \in \mathbb{R}^n. \tag{71}$$

Here

$$\mathscr{S}(F,\varphi)=\{(x_1,0): K_0x_1\in-\partial\chi(0)\}$$

and

$$\mathscr{E}(F,\varphi,V)=\{x\in\mathbb{R}^n: x_2^{\mathrm{T}}R^{\mathrm{T}}CRx_2+\chi(x_2)-\chi(0)=0\}=\{(x_1,0);x_1\in\mathbb{R}^m\}.$$

Moreover, it can be proved that the largest invariant subset $\mathscr{M}(F,\varphi,V)$ of $\mathscr{E}(F,\varphi,V)$ coincides with the set of stationary solutions of (19) (see the proof of Theorem 9 in [1]), that is

$$\mathscr{M}(F,\varphi,V)=\mathscr{S}(F,\varphi).$$

Our theory can be applied provided that the trivial stationary solution of (68) is isolated in $\mathscr{S}(F,\varphi)$. Let us so now assume in place of (H_4) that

$(H_4)'$ $\partial\Phi(0)=\{0\}$.

Then $\mathscr{S}(F,\varphi)=\{0\}$. So, if assumptions (H_1)–(H_3), $(H_4)'$ are satisfied, then properties (69)–(71) hold, $\mathscr{M}(F,\varphi,V)=\{0\}$ and we may apply Theorem 9 to conclude that if

$$\det\ K_0<0$$

then the trivial stationary solution of (68) is unstable.

11. Uncited references

[5,32].

Appendix A.

Proof of Claim 1. We first prove that there exists $T^*>0$ such that

$$\|S(T^*)P_{D(\partial\varphi)}x\|\leqslant\frac{r}{2},\quad\forall x\in\overline{B}_r.$$

Indeed, the stability of 0 as a solution of (19) ensures the existence of $\alpha>0$ such that if $\|S(\bar{t})P_{D(\partial\varphi)}x\|\leqslant\alpha$ for some $\bar{t}>0$ then $\|S(t)P_{D(\partial\varphi)}x\|\leqslant r/2$, $\forall t\geqslant\bar{t}$. Let $X\in\overline{B}_r$ be given. The attractivity of 0 as a solution of (19) ensures the existence of $T:=T(X)>0$ such that $\|S(T)P_{D(\partial\varphi)}X\|\leqslant\alpha/2$. Setting $\sigma:=(\alpha/2)\sqrt{\mathrm{e}^{-2\bar{\omega}T}}$, we see that, for $v\in B(X,\sigma):=\{z\in\mathbb{R}^n:\|z-X\|<\sigma\}$, we have (see (28))

$$\begin{aligned}\|S(T)P_{D(\partial\varphi)}v\|&\leqslant\|S(T)P_{D(\partial\varphi)}v-S(T)P_{D(\partial\varphi)}X\|+\|S(T)P_{D(\partial\varphi)}X\|\\&\leqslant\sqrt{\mathrm{e}^{2\bar{\omega}T}}\|P_{D(\partial\varphi)}v-P_{D(\partial\varphi)}X\|+\frac{\alpha}{2}\leqslant\sqrt{\mathrm{e}^{2\bar{\omega}T}}\|v-X\|+\frac{\alpha}{2}\leqslant\alpha.\end{aligned}$$

We see that

$$\overline{B}_r \subset \bigcup_{X\in\overline{B}_r} B(X,\sigma)$$

and since $\overline{B}_r$ is compact, we find $X_1, X_2, \ldots, X_N$ (and corresponding T_i, σ_i) such that

$$\overline{B}_r \subset \bigcup_{i=1}^{N} B(X_i,\sigma_i).$$

Setting $T^* := \max\{T_1, T_2, \ldots, T_N\}$, we see that if $x \in \overline{B}_r$ then $x \in B(X_i,\sigma_i)$ for some $i \in \{1,\ldots,N\}$. Then $\|S(T_i)P_{D(\partial\varphi)}x\| \leqslant \alpha$ and thus $\|S(t)P_{D(\partial\varphi)}x\| \leqslant r/2$, $\forall t \geqslant T^* (\geqslant T_i)$.

Let us now check that

$$\deg(\mathrm{id}_{\mathbb{R}^n} - S(T^*)P_{D(\partial\varphi)}., B_r, 0) = 1.$$

Let us define the continuous homotopy $h : [0,1]\times\mathbb{R}^n \to \mathbb{R}^n$ by

$$h(\lambda, x) := x - \lambda S(T^*)P_{D(\partial\varphi)}x.$$

We claim that $h(\lambda,x) \neq 0$, $\forall\lambda\in[0,1]$, $x \in \partial B_r$. Indeed, suppose on the contrary that there exists $x \in \partial B_r$ and $\lambda \in [0,1]$ such that

$$x = \lambda S(T^*)P_{D(\partial\varphi)}x.$$

Here $x = \lambda S(T^*)P_{D(\partial\varphi)}x + (1-\lambda)0$ and thus $x \in D(\partial\varphi)$ since $S(T^*)P_{D(\partial\varphi)}x \in D(\partial\varphi)$, $0 \in D(\partial\varphi)$ (by assumption (h_3)), $\lambda\in[0,1]$ and $D(\partial\varphi)$ is convex. It results that $x = \lambda S(T^*)x$ and thus

$$\|x - \lambda S(T^*)x\| = 0.$$

However,

$$\|x - \lambda S(T^*)x\| \geqslant \|x\| - \lambda\|S(T^*)x\| \geqslant \|x\| - \|S(T^*)x\| \geqslant \frac{r}{2} > 0,$$

which is a contradiction. Thus from properties 2 and 3 (Section 2) we get

$$\begin{aligned}\deg(\mathrm{id}_{\mathbb{R}^n} - S(T^*)P_{D(\partial\varphi)}., B_r, 0) &= \deg(h(1,.)., B_r, 0)\\ &= \deg(h(0,.), B_r, 0) = \deg(\mathrm{id}_{\mathbb{R}^n}, B_r, 0) = 1.\end{aligned}$$

We end the proof of Claim 1 by remarking that

$$\deg(\mathrm{id}_{\mathbb{R}^n} - S(t)P_{D(\partial\varphi)}., B_r, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - S(T^*)P_{D(\partial\varphi)}., B_r, 0), \quad \forall t > 0.$$

Indeed, let $t > 0$ be given and define the continuous homotopy $g : [0,1]\times\mathbb{R}^n \to \mathbb{R}^n$ by

$$g(\lambda, x) := x - S(\lambda t + (1-\lambda)T^*)P_{D(\partial\varphi)}x.$$

We see that $g(\lambda,x) \neq 0$, $\forall\lambda\in[0,1]$, $x \in \partial B_r$. Indeed, suppose on the contrary that there exists $x \in \partial B_r$ and $\lambda \in [0,1]$ such that $x = S(\lambda t + (1-\lambda)T^*)P_{D(\partial\varphi)}x$. Then $x \in D(\partial\varphi)$ and thus

$$x = S(\lambda t + (1-\lambda)T^*)x.$$

That means that the application $\tau \to S(\tau)x$ is periodic with period $\lambda t + (1-\lambda)T^*$ (see Section 5) and nontrivial since $\|S(0)x\| = \|x\| = r > 0$. This is a contradiction to (40). Thus from property 2 in Section 2

$$\begin{aligned}\deg(\mathrm{id}_{\mathbb{R}^n} - S(t)P_{D(\partial\varphi)}., B_r, 0) &= \deg(g(1,.), B_r, 0)\\ &= \deg(g(0,.), B_r, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - S(T^*)P_{D(\partial\varphi)}., B_r, 0). \qquad \square\end{aligned}$$

Proof of Claim 2. Let $t > 0$ be given and define the homotopy $H : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$H(\lambda, x) := x - (\lambda \mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1}(\lambda \mathrm{id}_{\mathbb{R}^n} + tG)P_{D(\partial\varphi)}x,$$

where G is defined in (42). We first remark that H is continuous. Indeed, let $x_n \to x$ and $\lambda_n \to \lambda$ and set $f_n := (\lambda_n \mathrm{id}_{\mathbb{R}^n} + tG)P_{D(\partial\varphi)}x_n$ and $v_n := (\lambda_n \mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1} f_n$. It is clear that

$$f_n \to f := (\lambda \mathrm{id}_{\mathbb{R}^n} + tG)P_{D(\partial\varphi)}x.$$

We claim that

$$v_n \to v := (\lambda \mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1} f.$$

Indeed, we have

$$\begin{aligned}v_n := (\lambda_n \mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1} f_n &\Leftrightarrow \frac{f_n - \lambda_n v_n}{t} \in \mathscr{A}_\varphi v_n\\ &\Leftrightarrow \left\langle v_n - \left(\frac{f_n - \lambda_n v_n}{t}\right), w - v_n \right\rangle + \varphi(w) - \varphi(v_n) \geqslant 0, \quad \forall w \in \mathbb{R}^n. \qquad (72)\end{aligned}$$

Setting $w = 0$ in (72), we get

$$\left\langle v_n - \left(\frac{f_n - \lambda_n v_n}{t}\right), v_n \right\rangle \leqslant \varphi(0) - \varphi(v_n)$$

and then using (29), we obtain

$$\begin{aligned}(t + \lambda_n)\|v_n\|^2 &\leqslant \|f_n\|\|v_n\| + t[\langle -F(0), v_n\rangle + \varphi(0) - \varphi(v_n)] + t\langle F(0), v_n\rangle\\ &\leqslant \|f_n\|\|v_n\| + t\|F(0)\|\|v_n\|.\end{aligned}$$

It results that the sequence $\{v_n\}$ is bounded.

Setting $w = v$ in (72), we get

$$\langle tv_n + \lambda_n v_n - f_n, v - v_n\rangle + t(\varphi(v) - \varphi(v_n)) \geqslant 0. \qquad (73)$$

We remark also that $v = (\lambda \mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1} f$ if and only if

$$\left\langle v - \left(\frac{f - \lambda v}{t}\right), w - v \right\rangle + \varphi(w) - \varphi(v) \geqslant 0, \quad \forall w \in \mathbb{R}^n,$$

and thus

$$\langle tv + \lambda v - f, v_n - v\rangle + t(\varphi(v_n) - \varphi(v)) \geqslant 0. \qquad (74)$$

So, from (73)

$$\langle tv_n + \lambda_n v_n, v_n - v\rangle \leqslant t(\varphi(v) - \varphi(v_n)) + \langle f_n, v_n - v\rangle$$

and from (74)

$$-\langle tv + \lambda v, v_n - v\rangle \leqslant t(\varphi(v_n) - \varphi(v)) - \langle f, v_n - v\rangle.$$

Thus

$$\langle t(v_n - v) + \lambda_n v_n - \lambda v, v_n - v\rangle \leqslant \langle f_n - f, v_n - v\rangle.$$

We obtain

$$(t + \lambda)\|v_n - v\|^2 \leqslant |\lambda - \lambda_n|\ \|v_n\|\ \|v_n - v\| + \|f_n - f\|\ \|v_n - v\|.$$

The sequence $\{v_n\}$ is bounded and thus $\{v_n\} \subset \overline{B}_K$ for some constant $K > 0$. It results that

$$\|v_n - v\| \leqslant K|\lambda - \lambda_n| + \|f_n - f\|$$

and thus $v_n \to v$ as $n \to \infty$.

We check now that $H(\lambda, x) \neq 0, \forall \lambda \in [0, 1], x \in \partial B_r$. Suppose on the contrary that there exists $x \in \partial B_r$ and $\lambda \in [0, 1]$ such that

$$x = (\lambda \mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1}(\lambda \mathrm{id}_{\mathbb{R}^n} + tG)P_{D(\partial\varphi)}x.$$

Then

$$\frac{\lambda(P_{D(\partial\varphi)}x - x) + tG(P_{D(\partial\varphi)}x)}{t} \in \mathscr{A}_\varphi(x).$$

Recalling that $\mathscr{A}_\varphi^{-1} = P_\varphi$, we get

$$x = P_\varphi\left(\frac{\lambda(P_{D(\partial\varphi)}x - x) + tG(P_{D(\partial\varphi)}x)}{t}\right). \tag{75}$$

Thus from (8) $x \in D(\partial\varphi)$ and (75) reduces to

$$x = P_\varphi(G(x)),$$

which is a contradiction to (38).

Then

$$\begin{aligned}\deg(\mathrm{id}_{\mathbb{R}^n} - (\mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1}(\mathrm{id}_{\mathbb{R}^n} + tG)P_{D(\partial\varphi)}., B_r, 0) &= \deg(H(1, .), B_r, 0)\\ &= \deg(H(0, .), B_r, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - (t\mathscr{A}_\varphi)^{-1}(tG(P_{D(\partial\varphi)}.)), B_r, 0)\\ &= \deg(\mathrm{id}_{\mathbb{R}^n} - P_\varphi G(P_{D(\partial\varphi)}.), B_r, 0). \qquad \square\end{aligned}$$

Proof of Claim 3. Let us consider the continuous homotopy $\mathscr{G} : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathscr{G}(\lambda, x) := x - P_\varphi(G(\lambda x + (1 - \lambda)P_{D(\partial\varphi)}x).$$

We claim that $\mathcal{G}(\lambda, x) \neq 0, \forall \lambda \in [0, 1], x \in \partial B_r$. Suppose on the contrary that there exists $x \in \partial B_r$ and $\lambda \in [0, 1]$ such that

$$x = P_\varphi G(\lambda x + (1-\lambda) P_{D(\partial\varphi)} x).$$

Then $x \in D(\partial\varphi)$ and thus

$$x = P_\varphi G(x),$$

which is a contradiction to (38). Thus

$$\begin{aligned}\deg(\mathrm{id}_{\mathbb{R}^n} - P_\varphi G(P_{D(\partial\varphi)}.), B_r, 0) &= \deg(\mathcal{G}(0, .), B_r, 0)\\ &= \deg(\mathcal{G}(1, .), B_r, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - P_\varphi G(.), B_r, 0). \quad \square\end{aligned}$$

Proof of Claim 4. (a) Recalling that $P_{D(\partial\varphi)} 0 = 0$, we obtain

$$\|P_{D(\partial\varphi)} x\| = \|P_{D(\partial\varphi)} x - P_{D(\partial\varphi)} 0\| \leqslant \|x\| \leqslant r \leqslant r_0.$$

(b) Recalling that $P_\varphi G(0) = 0$ (see (29) and (42)), we get

$$\begin{aligned}\|P_\varphi G(P_{D(\partial\varphi)} x)\| &= \|P_\varphi G(P_{D(\partial\varphi)} x) - P_\varphi G(0)\| \leqslant \|G(P_{D(\partial\varphi)} x) - G(0)\|\\ &\leqslant \|P_{D(\partial\varphi)} x\| + \|F(P_{D(\partial\varphi)} x) - F(0)\| \leqslant r_0(1 + C_0) \leqslant \frac{\delta_1}{2} < \delta_1.\end{aligned}$$

(c) This follows from (a) and (39).

(d) We know that

$$u(0) = P_{D(\partial\varphi)} x \tag{76}$$

and

$$\left\langle \frac{\mathrm{d}u}{\mathrm{d}s}(s), w - u(s) \right\rangle + \langle F(u(s)), w - u(s)\rangle + \varphi(w) - \varphi(u(s)) \geqslant 0, \quad \text{a.e. } s \geqslant 0. \tag{77}$$

Setting $w = P_{D(\partial\varphi)} x$ in (77), we obtain

$$\begin{aligned}\left\langle \frac{\mathrm{d}}{\mathrm{d}s}(P_{D(\partial\varphi)} x - u(s)), P_{D(\partial\varphi)} x - u(s) \right\rangle &\leqslant \langle F(u(s)), P_{D(\partial\varphi)} x - u(s)\rangle\\ &\quad + \varphi(P_{D(\partial\varphi)} x) - \varphi(u(s)), \quad \text{a.e. } s \geqslant 0.\end{aligned}$$

Using (a), (c) and (37), we obtain

$$\varphi(P_{D(\partial\varphi)} x) - \varphi(u(s)) \leqslant C_1 \|P_{D(\partial\varphi)} x - u(s)\| \leqslant C_1(r_0 + 1).$$

We have also

$$\begin{aligned}&\langle F(u(s)), P_{D(\partial\varphi)} x - u(s)\rangle\\ &\quad = \langle F(u(s)) - F(P_{D(\partial\varphi)} x), P_{D(\partial\varphi)} x - u(s)\rangle + \langle F(P_{D(\partial\varphi)} x), P_{D(\partial\varphi)} x - u(s)\rangle\\ &\quad \leqslant \bar{\omega}\|P_{D(\partial\varphi)} x - u(s)\|^2 + \|F(P_{D(\partial\varphi)} x)\|\|P_{D(\partial\varphi)} x - u(s)\|\\ &\quad \leqslant \bar{\omega}(r_0 + 1)^2 + (r_0 + 1)(\|F(P_{D(\partial\varphi)} x) - F(0)\| + \|F(0)\|)\\ &\quad \leqslant \bar{\omega}(r_0 + 1)^2 + (r_0 + 1)(C_0 r_0 + \|F(0)\|).\end{aligned}$$

Thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|P_{D(\partial\varphi)}x-u(s)\|^2\leqslant(r_0+1)(C_1+\bar{\omega}(r_0+1)+C_0r_0+\|F(0)\|),\quad \text{a.e. } s\geqslant 0.$$

Consequently, for $t>0$, we have

$$\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}\|P_{D(\partial\varphi)}x-u(s)\|^2\,\mathrm{d}s\leqslant 2(r_0+1)(C_1+\bar{\omega}(r_0+1)+C_0r_0+\|F(0)\|)t.$$

Using now (76), we obtain

$$\|P_{D(\partial\varphi)}x-u(t)\|\leqslant\theta\sqrt{t}.$$

(e) Using (c) and recalling that $P_\varphi G(0)=0$, we get

$$\begin{aligned}\|P_\varphi G(u(t))\|&=\|P_\varphi G(u(t))-P_\varphi G(0)\|\\&\leqslant\|G(u(t))-G(0)\|\leqslant\|u(t)\|+\|F(u(t))-F(0)\|\\&\leqslant\|u(t)\|(1+C_0)\leqslant\frac{\delta_1}{2}<\delta_1.\end{aligned}$$

(f) Let $t\in(0,t^*]$ be given. Using Theorem 4, we obtain

$$\|S(t^*)P_{D(\partial\varphi)}x-S(t^*)0\|\leqslant\sqrt{\mathrm{e}^{2w(t^*-t)}}\|S(t)P_{D(\partial\varphi)}x-S(t)0\|.$$

Thus

$$\|u(t)\|\geqslant\sqrt{\mathrm{e}^{2\bar{\omega}t}}\sqrt{\mathrm{e}^{-2\bar{\omega}t^*}}\|u(t^*)\|\geqslant\sqrt{\mathrm{e}^{-2\bar{\omega}t^*}}\frac{r}{4}\geqslant\frac{r}{8}.$$

(g) From the definition of $v(t)$, we see that

$$\frac{P_{D(\partial\varphi)}x-v(t)+tG(P_{D(\partial\varphi)}x)}{t}\in\mathscr{A}_\varphi(v(t)).$$

Thus

$$\left\langle\frac{v(t)-P_{D(\partial\varphi)}x}{t},w-v(t)\right\rangle+\langle v(t)-G(P_{D(\partial\varphi)}x),w-v(t)\rangle+\varphi(w)-\varphi(v(t))\geqslant 0,\quad\forall w\in\mathbb{R}^n.$$

It results also that

$$\frac{v(t)-P_{D(\partial\varphi)}x}{t}+v(t)-G(P_{D(\partial\varphi)}x)\in-\partial\varphi(v(t))$$

and thus $v(t)\in D(\partial\varphi)$.

(h) Applying the result in (g) with $w:=P_{D(\partial\varphi)}x$, we obtain

$$\frac{1}{t}\langle v(t)-P_{D(\partial\varphi)}x,P_{D(\partial\varphi)}x-v(t)\rangle+\langle v(t)-G(P_{D(\partial\varphi)}x),P_{D(\partial\varphi)}x-v(t)\rangle+\varphi(P_{D(\partial\varphi)}x)-\varphi(v(t))\geqslant 0.$$

Thus

$$\begin{aligned}
&\frac{1}{t}\|v(t) - P_{D(\partial\varphi)}x\|^2 + \|v(t) - P_{D(\partial\varphi)}x\|^2 \\
&\leqslant \langle P_{D(\partial\varphi)}x - G(P_{D(\partial\varphi)}x),\, P_{D(\partial\varphi)}x - v(t)\rangle + \varphi(P_{D(\partial\varphi)}x) - \varphi(v(t)) \\
&= \langle F(P_{D(\partial\varphi)}x),\, P_{D(\partial\varphi)}x - v(t)\rangle + \varphi(P_{D(\partial\varphi)}x) - \varphi(v(t)) \\
&= \langle F(P_{D(\partial\varphi)}x) - F(0),\, P_{D(\partial\varphi)}x - v(t)\rangle + \langle F(0),\, P_{D(\partial\varphi)}x\rangle - \langle F(0),\, v(t)\rangle \\
&\quad + \varphi(0) - \varphi(v(t)) + \varphi(P_{D(\partial\varphi)}x) - \varphi(0) \\
&\leqslant \|F(P_{D(\partial\varphi)}x) - F(0)\|\,\|P_{D(\partial\varphi)}x - v(t)\| + \|F(0)\|\,\|P_{D(\partial\varphi)}x\| \\
&\quad + |\varphi(P_{D(\partial\varphi)}x) - \varphi(0)| \\
&\leqslant C_0 r_0 \|P_{D(\partial\varphi)}x - v(t)\| + \|F(0)\| r_0 + C_1 r_0 \\
&\leqslant \frac{(C_0 r_0)^2}{2} + \frac{\|P_{D(\partial\varphi)}x - v(t)\|^2}{2} + \|F(0)\| r_0 + C_1 r_0.
\end{aligned}$$

Thus

$$\frac{1}{t}\|v(t) - P_{D(\partial\varphi)}x\|^2 \leqslant \frac{(C_0 r_0)^2}{2} + \|F(0)\| r_0 + C_1 r_0.$$

Therefore

$$\|v(t) - P_{D(\partial\varphi)}x\| \leqslant \overline{\theta}\sqrt{t}.$$

(i) Let $t \in (0, T^*]$ be given. From (a) and (h), we deduce that

$$\|v(t)\| \leqslant \overline{\theta}\sqrt{t} + r_0.$$

Thus

$$\|v(t)\| \leqslant \frac{\delta_0}{2} + r_0 \leqslant \delta_0$$

and

$$\|v(t)\| \leqslant \frac{\delta_1}{2(1 + C_0)} + r_0 \leqslant \frac{\delta_1}{1 + C_0}.$$

(j) Let $t \in (0, T^*]$ be given. Using (i), we get

$$\begin{aligned}
\|P_\varphi(G(v(t)))\| &= \|P_\varphi(G(v(t))) - P_\varphi(G(0))\| \leqslant \|G(v(t)) - G(0)\| \\
&\leqslant \|v(t)\| + \|F(v(t)) - F(0)\| \leqslant (1 + C_0)\|v(t)\| \leqslant \delta_1. \qquad \square
\end{aligned}$$

Proof of Claim 5. Let us define the homotopy $W : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$W(\lambda, x) := x - \lambda(\mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1}(\mathrm{id}_{\mathbb{R}^n} + tG)P_{D(\partial\varphi)}x - (1 - \lambda)S(t)P_{D(\partial\varphi)}x.$$

Note that $W(\lambda, x) = x - \lambda H(1, x) - (1 - \lambda)S(t)P_{D(\partial\varphi)}x$ where H is defined in Claim 2. The continuity of W follows from the continuity of the Poincaré operator $S(t)$ and the

continuity of H (see Claim 2). We claim that $W(\lambda, x) \neq 0, \forall \lambda \in [0,1], x \in \partial B_r$. Indeed, suppose on the contrary that there exists $x \in \partial B_r$ and $\lambda \in [0,1]$ such that

$$x = \lambda(\mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1}(\mathrm{id}_{\mathbb{R}^n} + tG)P_{D(\partial\varphi)}x + (1-\lambda)S(t)P_{D(\partial\varphi)}x.$$

Setting

$$u(t) := S(t)P_{D(\partial\varphi)}x$$

and

$$v(t) := (\mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1}(\mathrm{id}_{\mathbb{R}^n} + tG)P_{D(\partial\varphi)}x,$$

we may write

$$x = \lambda v(t) + (1-\lambda)u(t).$$

The following calculations will show that this equality is impossible to satisfy on the time interval $(0, \bar{T}]$. Here $u(t) \in D(\partial\varphi)$ and $v(t) \in D(\partial\varphi)$ (see Claim 4(g)). Thus $x \in D(\partial\varphi)$ since $D(\partial\varphi)$ is convex. It results that

$$P_{D(\partial\varphi)}x = x.$$

Moreover, from Claims 1 and 2, we deduce that $\lambda \in (0,1)$. We know that

$$\left\langle \frac{\mathrm{d}u}{\mathrm{d}s}(s), w - u(s) \right\rangle + \langle F(u(s)), w - u(s)\rangle + \varphi(w) - \varphi(u(s)) \geqslant 0,$$
$$\forall w \in \mathbb{R}^n, \quad \text{a.e. } s \geqslant 0, \tag{78}$$

and thus

$$\int_0^t \left\langle \frac{\mathrm{d}u}{\mathrm{d}s}(s), w - u(s) \right\rangle \mathrm{d}s + \int_0^t \varphi(w)\,\mathrm{d}s - \int_0^t \varphi(u(s))\,\mathrm{d}s$$
$$\geqslant -\int_0^t \langle F(u(s)), w - u(s)\rangle\,\mathrm{d}s, \quad \forall w \in \mathbb{R}^n.$$

On the other hand

$$\begin{aligned}\int_0^t \left\langle \frac{\mathrm{d}u}{\mathrm{d}s}(s), u(s) - x \right\rangle \mathrm{d}s &= \int_0^t \left\langle \frac{\mathrm{d}}{\mathrm{d}s}(u(s) - x), u(s) - x \right\rangle \mathrm{d}s \\ &= \frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\|u(s) - x\|^2\,\mathrm{d}s \\ &= \frac{1}{2}\|u(t) - x\|^2 - \frac{1}{2}\|u(0) - x\|^2 \\ &= \frac{1}{2}\|u(t) - x\|^2 \geqslant 0.\end{aligned}$$

Thus

$$\int_0^t \left\langle \frac{\mathrm{d}u}{\mathrm{d}s}(s), w - x \right\rangle \mathrm{d}s \geqslant \int_0^t \varphi(u(s))\,\mathrm{d}s - \int_0^t \varphi(w)\,\mathrm{d}s - \int_0^t \langle u(s) - G(u(s)),$$
$$\times\, w - u(s)\rangle\,\mathrm{d}s, \quad \forall w \in \mathbb{R}^n. \tag{79}$$

1 Let us set

$$\Xi(t) := \int_0^t \left\langle \frac{\mathrm{d}u}{\mathrm{d}s}(s),\, P_\varphi(G(x)) - x \right\rangle \mathrm{d}s.$$

3 Setting $w := P_\varphi(G(x))$ in (79), we see that

$$\begin{aligned}\Xi(t) &\geqslant \int_0^t \varphi(u(s))\,\mathrm{d}s - \int_0^t \varphi(P_\varphi(G(x)))\,\mathrm{d}s - \int_0^t \langle u(s) - G(u(s)),\, P_\varphi(G(x)) \\ &\quad - u(s)\rangle\,\mathrm{d}s \\ &= \int_0^t \varphi(u(s))\,\mathrm{d}s - \int_0^t \varphi(P_\varphi(G(x)))\,\mathrm{d}s - \int_0^t \langle u(s) - G(u(s)),\, P_\varphi(G(u(s))) \\ &\quad - u(s)\rangle\,\mathrm{d}s - \int_0^t \langle u(s) - G(u(s)),\, P_\varphi(G(x)) - P_\varphi(G(u(s)))\rangle\,\mathrm{d}s.\end{aligned}$$

5 By definition of P_φ, we have

$$\langle P_\varphi(G(u(s))) - G(u(s)),\, w - P_\varphi(G(u(s)))\rangle + \varphi(w) - \varphi(P_\varphi(G(u(s)))) \geqslant 0,$$
$$\forall w \in \mathbb{R}^n.$$

7 Setting $w := u(s)$, we get

$$\begin{aligned}\langle P_\varphi(G(u(s))),\, G(u(s))\rangle \geqslant &- \langle P_\varphi(G(u(s))),\, u(s)\rangle + \langle G(u(s)),\, u(s)\rangle \\ &+ \|P_\varphi(G(u(s)))\|^2 + \varphi(P_\varphi(G(u(s)))) - \varphi(u(s)).\end{aligned}$$

9 Thus

$$\begin{aligned}&\varphi(u(s)) - \varphi(P_\varphi(G(x))) + \langle u(s) - G(u(s)),\, u(s) - P_\varphi(G(u(s)))\rangle \\ &\quad = \varphi(u(s)) - \varphi(P_\varphi(G(x))) + \|u(s)\|^2 + \langle G(u(s)),\, P_\varphi(G(u(s)))\rangle \\ &\qquad - \langle G(u(s)),\, u(s)\rangle - \langle u(s),\, P_\varphi(G(u(s)))\rangle \\ &\quad \geqslant \varphi(P_\varphi(G(u(s)))) - \varphi(P_\varphi(G(x))) + \|u(s)\|^2 - 2\langle P_\varphi(G(u(s))),\, u(s)\rangle \\ &\qquad + \|P_\varphi(G(u(s)))\|^2 \\ &\quad = \varphi(P_\varphi(G(u(s)))) - \varphi(P_\varphi(G(x))) + \|u(s) - P_\varphi(G(u(s)))\|^2.\end{aligned}$$

11 Thus

$$\begin{aligned}\Xi(t) \geqslant &\int_0^t \varphi(P_\varphi(G(u(s)))) - \varphi(P_\varphi(G(x))) + \|u(s) - P_\varphi(G(u(s)))\|^2\,\mathrm{d}s \\ &- \int_0^t \langle u(s) - G(u(s)),\, P_\varphi(G(x)) - P_\varphi(G(u(s)))\rangle\,\mathrm{d}s.\end{aligned}$$

13 Using Claim 4(e) and (a)–(d), we see that

$$\begin{aligned}&\varphi(P_\varphi(G(u(s)))) - \varphi(P_\varphi(G(x))) \leqslant C_1\|P_\varphi(G(u(s))) - P_\varphi(G(x))\| \\ &\quad \leqslant C_1\|G(u(s)) - G(x)\| \leqslant C_1\|u(s) - x\| + C_1\|F(u(s)) - F(x)\| \\ &\quad \leqslant \theta(C_1 + C_1C_0)\sqrt{s}.\end{aligned}$$

Using Claim 4(c), (a) and (d), we obtain also

$$\begin{aligned}&\langle u(s) - G(u(s)),\, P_\varphi(G(x)) - P_\varphi(G(u(s)))\rangle\\ &\quad\leqslant \|F(u(s))\|\,\|P_\varphi(G(x)) - P_\varphi(G(u(s)))\|\\ &\quad\leqslant (\|F(0)\| + C_0\|u(s)\|)\|G(x) - G(u(s))\|\\ &\quad\leqslant (\|F(0)\| + C_0)(\|x - u(s)\| + \|F(x) - F(u(s))\|)\\ &\quad\leqslant (\|F(0)\| + C_0)(1 + C_0)\|x - u(s)\| \leqslant (\|F(0)\| + C_0)(1 + C_0)\theta\sqrt{s}.\end{aligned}$$

Thus

$$\Xi(t) \geqslant \int_0^t \|u(s) - P_\varphi(G(u(s)))\|^2\,\mathrm{d}s - \frac{2}{3}\theta(1 + C_0)(\|F(0)\| + C_0 + C_1)t^{3/2}.$$

If $\|u(t)\| \geqslant r/4$ then using Claim 4(f), we may assert that $\|u(s)\| \geqslant r/8$, $\forall s \in (0, t]$ and then using (41), we obtain that

$$\Xi(t) \geqslant \eta^2 t - Ct^{3/2} \quad \text{if } \|u(t)\| \geqslant \frac{r}{4}. \tag{80}$$

It is also clear that

$$\Xi(t) \geqslant -Ct^{3/2} \quad \text{if } \|u(t)\| < \frac{r}{4}. \tag{81}$$

By Claim 4(g), we know that

$$\left\langle \frac{v(t) - x}{t}, w - v(t)\right\rangle + \langle v(t) - G(x), w - v(t)\rangle + \varphi(w) - \varphi(v(t)) \geqslant 0,$$
$$\forall w \in \mathbb{R}^n.$$

Moreover,

$$\begin{aligned}\left\langle \frac{v(t) - x}{t}, w - v(t)\right\rangle &= \left\langle \frac{v(t) - x}{t}, w - x\right\rangle + \left\langle \frac{v(t) - x}{t}, x - v(t)\right\rangle\\ &\leqslant \left\langle \frac{v(t) - x}{t}, w - x\right\rangle = -\frac{(1 - \lambda)}{\lambda t}\langle u(t) - x, w - x\rangle\\ &= -\frac{(1 - \lambda)}{\lambda t}\int_0^t \left\langle \frac{\mathrm{d}u}{\mathrm{d}s}(s), w - x\right\rangle \mathrm{d}s.\end{aligned}$$

Setting $v := (1 - \lambda)/\lambda$, we see that

$$-\frac{v}{t}\int_0^t \left\langle \frac{\mathrm{d}u}{\mathrm{d}s}(s), w - x\right\rangle \mathrm{d}s + \langle v(t) - G(x), w - v(t)\rangle + \varphi(w) - \varphi(v(t)) \geqslant 0,$$
$$\forall w \in \mathbb{R}^n.$$

Setting $w := P_\varphi(G(x))$, we get

$$\begin{aligned}\Xi(t) &\leqslant \frac{t}{v}[\langle v(t) - G(x), P_\varphi(G(x)) - v(t)\rangle + \varphi(P_\varphi(G(x))) - \varphi(v(t))]\\ &= \frac{t}{v}[\langle v(t) - G(v(t)), P_\varphi(G(x)) - v(t)\rangle + \langle G(v(t)) - G(x), P_\varphi(G(x))\\ &\quad - v(t)\rangle + \varphi(P_\varphi(G(x))) - \varphi(v(t))]\\ &= \frac{t}{v}[\langle v(t) - G(v(t)), P_\varphi(G(v(t))) - v(t)\rangle + \varphi(P_\varphi(G(v(t)))) - \varphi(v(t))\\ &\quad + \langle v(t) - G(v(t)), P_\varphi(G(x)) - P_\varphi(G(v(t)))\rangle + \langle G(v(t))\\ &\quad - G(x), P_\varphi(G(x)) - v(t)\rangle + \varphi(P_\varphi(G(x))) - \varphi(P_\varphi(G(v(t))))].\end{aligned}$$

We know that

$$\langle P_\varphi(G(v(t))) - G(v(t)), w - P_\varphi(G(v(t)))\rangle + \varphi(w) - \varphi(P_\varphi(G(v(t)))) \geqslant 0, \quad \forall w \in \mathbb{R}^n.$$

Setting $w := v(t)$, we see that

$$\langle P_\varphi(G(v(t))) - G(v(t)), P_\varphi(G(v(t))) - v\rangle + \varphi(P_\varphi(G(v(t)))) - \varphi(v(t)) \leqslant 0$$

and thus

$$\begin{aligned}&\langle v(t) - G(v(t)), P_\varphi(G(v(t))) - v(t)\rangle + \varphi(P_\varphi(G(v(t)))) - \varphi(v(t))\\ &\quad \leqslant -\|P_\varphi(G(v(t))) - v(t)\|^2.\end{aligned}$$

Thus

$$\begin{aligned}\Xi(t) \leqslant &-\frac{t}{v}\|P_\varphi(G(v(t))) - v(t)\|^2 + \frac{t}{v}[\langle v(t) - G(v(t)), P_\varphi(G(x))\\ &- P_\varphi(G(v(t)))\rangle + \langle G(v(t)) - G(x), P_\varphi(G(x)) - v(t)\rangle\\ &+ \varphi(P_\varphi(G(x))) - \varphi(P_\varphi(G(v(t))))].\end{aligned}$$

Using Claim 4(a), (h) and (i), we see that

$$\begin{aligned}&\langle v(t) - G(v(t)), P_\varphi(G(x)) - P_\varphi(G(v(t)))\rangle\\ &\quad \leqslant \|F(v(t))\|\|G(x) - G(v(t))\|\\ &\quad \leqslant (\|F(0)\| + C_0\delta_0)(\|x - v(t)\| + \|F(x) - F(v(t))\|)\\ &\quad \leqslant (\|F(0)\| + C_0\delta_0)(1 + C_0)\overline{\theta}\sqrt{t}.\end{aligned}$$

Using Claim 4(a), (b) and (i), we obtain also that

$$\begin{aligned}&\langle G(v(t)) - G(x), P_\varphi(G(x)) - v(t)\rangle\\ &\quad \leqslant (\|P_\varphi(G(x))\| + \|v(t)\|)(\|x - v(t)\| + \|F(x) - F(v(t))\|)\\ &\quad \leqslant (\delta_1 + \delta_0)(1 + C_0)\overline{\theta}\sqrt{t}.\end{aligned}$$

Moreover, using Claim 4(b), (g), (a), (i) and (h), we get

$$\begin{aligned}&|\varphi(P_\varphi(G(x))) - \varphi(P_\varphi(G(v(t))))| \leqslant C_1\|P_\varphi(G(x)) - P_\varphi(G(v(t)))\|\\ &\quad \leqslant C_1\|G(x) - G(v(t))\| \leqslant C_1(1 + C_0)\|x - v(t)\| \leqslant C_1(1 + C_0)\overline{\theta}\sqrt{t}.\end{aligned}$$

Thus

$$\begin{aligned}\Xi(t) &\leqslant -\frac{t}{v}\|P_{\varphi}(G(v(t))) - v(t)\|^2 + \frac{1}{v}t^{3/2}[\overline{\theta}(1+C_0)(\|F(0)\| + \delta_1 \\ &\quad + (1+C_0)\delta_0 + C_1)] \\ &\leqslant -\frac{t}{v}\|P_{\varphi}(G(v(t))) - v(t)\|^2 + \frac{C}{v}t^{3/2}\end{aligned}$$

Thus

$$\Xi(t) \leqslant -\eta^2\frac{t}{v} + \frac{C}{v}t^{3/2} \quad \text{if } \|v(t)\| \geqslant \frac{r}{8} \tag{82}$$

and

$$\Xi(t) \leqslant \frac{C}{v}t^{3/2} \quad \text{if } \|v(t)\| < \frac{r}{8}. \tag{83}$$

If $\|u(t)\| \geqslant r/4$ and $\|v(t)\| \geqslant r/8$ then from (80) and (82), we get

$$\eta^2 t - Ct^{3/2} \leqslant -\eta^2\frac{t}{v} + \frac{C}{v}t^{3/2}$$

and thus $t \geqslant \eta^4/C^2$ which is a contradiction since $t \leqslant \overline{T} \leqslant \frac{1}{2}\eta^4/C^2$.

If $\|u(t)\| < r/4$ and $\|v(t)\| \geqslant r/8$ then from (81) and (82), we get

$$-Ct^{3/2} \leqslant -\eta^2\frac{t}{v} + \frac{C}{v}t^{3/2}. \tag{84}$$

Using Claim 4(i), we see that

$$v = \frac{\|v(t) - x\|}{\|x - u(t)\|} \leqslant \frac{\|v(t)\| + \|x\|}{\|x\| - \|u(t)\|} \leqslant v^*,$$

where $v^* = \frac{4}{3}(\delta_0 + r)/r$. Here (84) yields $t \geqslant \eta^4/C^2(v^*+1)^2$ which contradicts $t \leqslant \bar{T}$.

If $\|u(t)\| \geqslant r/4$ and $\|v(t)\| < r/8$ then from (80) and (83), we get

$$\eta^2 t - Ct^{3/2} \leqslant \frac{C}{v}t^{3/2}. \tag{85}$$

Using Claim 4(c), we see that

$$v = \frac{\|v(t) - x\|}{\|x - u(t)\|} \geqslant \frac{\|x\| - \|v(t)\|}{\|x\| + \|u(t)\|} \geqslant v_*,$$

where $v_* = 7r/8(1+r)$. Then (85) gives $t \geqslant \eta^4/C^2(1/v_* + 1)^2$ and a contradiction to $t \leqslant \bar{T}$.

If $\|u(t)\| < r/4$ and $\|v(t)\| < r/8$ then

$$\|x\| \leqslant \lambda\|v(t)\| + (1-\lambda)\|u(t)\| \leqslant \lambda\frac{r}{8} + (1-\lambda)\frac{r}{4} < r$$

which is a contradiction since $x \in \partial B_r$.

Consequently one has $x \neq \lambda v(t) + (1-\lambda)u(t), \forall x \in \partial B_r, \lambda \in [0,1]$ so that we have proved $W(\lambda, x) \neq 0, \forall x \in \partial B_r, \lambda \in [0,1]$. Thus

$$\deg(\mathrm{id}_{\mathbb{R}^n} - (\mathrm{id}_{\mathbb{R}^n} + t\mathscr{A}_\varphi)^{-1}(\mathrm{id}_{\mathbb{R}^n} + tG)P_{D(\partial\varphi)}., B_r, 0) = \deg(W(1,.), B_r, 0)$$
$$= \deg(W(0,.), B_r, 0) = \deg(\mathrm{id}_{\mathbb{R}^n} - S(t)P_{D(\partial\varphi)}., B_r, 0).$$

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