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Systems & Control Letters 55 (2006) 350–367
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

On the controllability of linear juggling mechanical systems

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Received 11 May 2004; received in revised form 31 January 2005; accepted 31 August 2005
Available online 18 October 2005

Abstract

This paper deals with the controllability of a class of nonsmooth complementarity mechanical systems. Due to their particular structure they can be decomposed into an "object" and a "robot", consequently they are named *juggling* systems. It is shown that the accessibility of the "object" can be characterized by nonlinear constrained equations, or generalized equations. Examples are presented, including a simple model of backlash. The main focus of the work is about linear jugglers.

Keywords: Unilateral constraints; Nonsmooth mechanics; Impacts; Controllability; Complementarity problems; Accessibility

1. Introduction

Manipulating objects by pushing and hitting (also called nonprehensile manipulation) is an important robotic task, see [1,16,12,24] and references therein. It is easily recast in the setting of so-called juggling systems [9,15,26,28] (a class that encompasses systems with dynamic backlash [18], manipulators with dynamic passive environments, controlled structures, hopping machines, tethered sattelites [14], etc.).

A powerful way to model many physical systems is to use the so-called complementarity formalism [11,5]. In particular, mechanical systems subject to unilateral constraints belong to this class of nonsmooth dynamical systems. From a general point of view, controllability, observability, and stabilizability of such systems have not yet been understood, except in particular cases [19,20,6,15,10,26–28]. This is typically the case for jugglers which form a subclass of complementarity dynamical systems.

Studies on the controllability of such nonlinear nonsmooth dynamical systems require the development of specific analysis tools, due to their very particular features [8,16,27,15]. The paper [15] contains a very nice study of a juggling system and its reachable subspaces, and proposes a general method for the design of feedback control in order to stabilize specific trajectories. In particular, the studied juggler is not small-time locally controllable. Global criteria for accessibility may be needed. This paper is dedicated to investigate a way to characterize the controllability properties of a subclass of juggling systems, which we choose to name *linear jugglers*. It appears that despite the fact this class of jugglers may represent the simplest juggling systems, their controllability is not easy to establish, in general, since they anyway remain highly nonlinear dynamical systems. This work presents some tools which allows one to characterize in a general way whether the considered system possesses the required accessibility properties, or not. The paper is organized as follows: in Section 2, we introduce the dynamics of jugglers; in Section 3 the controllability framework is developed; Section 4 is devoted to illustrating the theoretical setting by an example (dynamic backlash). Conclusions end the paper. Some definitions and calculations are provided in appendices. A preliminary version of this work can be found in [7].

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0167-6911/$ - see front matter
doi:10.1016/j.sysconle.2005.08.012

2. System's dynamics

2.1. Introduction

Let us consider the following class of complementarity dynamical systems [11,5]:

$$\begin{cases} \dot{z}_1 = f_1(z_1, t, \lambda), \\ \dot{z}_2 = f_2(z_2, t, u, \lambda), \\ 0 \leqslant h(z_1, z_2) \perp \lambda \geqslant 0, \\ \text{Collision mapping} \end{cases} \tag{1}$$

which has been named *juggling systems* in [8], where $z_1 \in R^{n_1}$, $z_2 \in R^{n_2}$, $h(\cdot, \cdot)$ and $\lambda \in R^m$, and $u \in R^{n_u}$. The z_1-dynamics represents the dynamics of the "object" (which may be a real object like a puck, or the center of gravity dynamics of a flying system [4, Section 8.7]), while the z_2-dynamics is that of the "robot". The signal λ in (1) is a vector of Lagrange multipliers which represents the contact force between the two parts of the system, if the system is a mechanical system. When the "distance" function $h(z_1, z_2)$ is positive, then the interaction is $\lambda = 0$, and the force is allowed to be $\lambda > 0$ only if $h(z_1, z_2) = 0$. At times of impact, λ is no longer a function but is a Dirac measure so that the dynamics becomes algebraic [4]. Obviously, the free-motion dynamics ($\lambda \equiv 0$) is not controllable. Though the system in (1) can model various physical systems with nonsmooth effects, we will focus in this paper on mechanical systems subject to unilateral constraints. In addition, only a subclass of dynamics as in (1) that we may call *linear jugglers* will be examined:

$$\begin{cases} M_1 \ddot{q}_1 = A_1^{\mathrm{T}} \lambda, \\ M_2 \ddot{q}_2 = Eu + A_2^{\mathrm{T}} \lambda, \\ 0 \leqslant Aq + B \perp \lambda \geqslant 0, \\ \dot{q}(t_k^+) = \mathrm{prox}_M[\dot{q}(t_k^-), V(q(t_k))]. \end{cases} \tag{2}$$

In (2) $q_1 \in \mathbb{R}^{n_1/2 \times 1}$, $q_2 \in \mathbb{R}^{n_2/2 \times 1}$, $q^{\mathrm{T}} = (q_1^{\mathrm{T}}, q_2^{\mathrm{T}})$ is a $\frac{n_1+n_2}{2}$-dimensional vector of generalized coordinates, $A = (A_1, A_2) \in \mathbb{R}^{m \times (n_1/2 + n_2/2)}$, i.e. A_1 is made of the first $n_1/2$ columns of A, whereas A_2 is made of the last $n_2/2$ columns of A. Also $E \in \mathbb{R}^{n_2/2 \times n_u}$, $M_1 \in \mathbb{R}^{n_1/2 \times n_1/2}$, $M_2 \in \mathbb{R}^{n_2/2 \times n_2/2}$ both full-rank, and $B \in \mathbb{R}^m$ are all constant, $\lambda \in \mathbb{R}^m$. Clearly, both n_1 and n_2 are even integers. The "prox_M" denotes the proximation in the kinetic metric, i.e. $\dot{q}(t_k^+)$ is the closest vector to $\dot{q}(t_k^-)$ inside the set $V(q(t_k))$, and with the distance deduced from the scalar product $x^{\mathrm{T}} My$, x and $y \in \mathbb{R}^{n_1/2 + n_2/2}$. The times t_k generically denote impact times.

Example 1. For the backlash model in Fig. 1, one has

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ L \end{pmatrix},$$

$m = 2$, $n_1 = n_2 = 2$. Thus,

$$A_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The controllability properties of this example will be examined in more detail in Section 4.

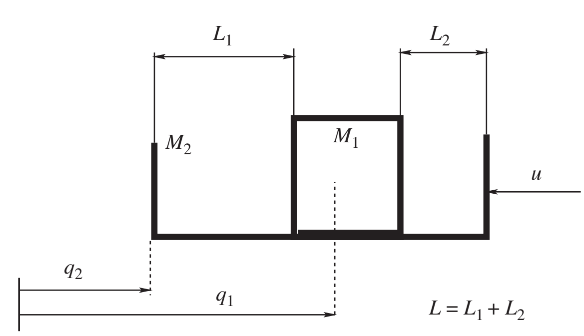

Fig. 1. The impacting pair.

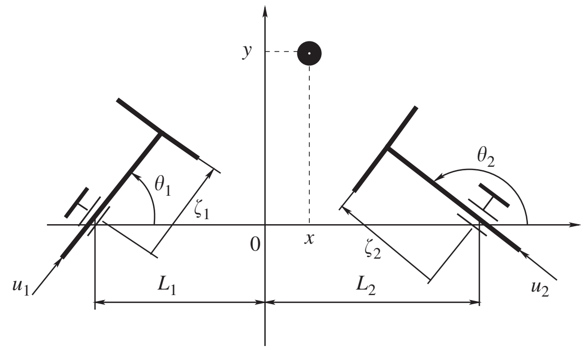

Fig. 2. A nonprehensile manipulation system.

Example 2. Let us consider the nonprehensile manipulation system without gravity depicted in Fig. 2. Then one has $n_1 = n_2 = 4$, $m = 2$,

$$A = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & -1 & 0 \\ \cos\theta_2 & \sin\theta_2 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} L_1\cos\theta_1 \\ -L_2\cos\theta_2 \end{pmatrix}, \quad E = I_2,$$

$$M_1 = \begin{pmatrix} m_3 & 0 \\ 0 & m_3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad q_1 = \begin{pmatrix} x \\ y \end{pmatrix}, \quad q_2 = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix},$$

where I_2 is the 2-by-2 identity matrix. We suppose that $0 < \theta_1 < \pi$, $0 < \theta_2 < \pi$.

At $t = t_k$ the multiplier $\lambda = p_k \delta_{t_k}$ is a Dirac measure, $p_k \geqslant 0$, and the dynamics in (2) can be rewritten as

$$\begin{cases} M_1(\dot{q}_1(t_k^+) - \dot{q}_1(t_k^-)) = A_1^{\mathrm{T}} p_k, \\ M_2(\dot{q}_2(t_k^+) - \dot{q}_2(t_k^-)) = A_2^{\mathrm{T}} p_k, \\ \dot{q}(t_k^+) = \mathrm{prox}_M[\dot{q}(t_k^-), V(q(t_k))]. \end{cases} \tag{3}$$

It is implicitly assumed in (2) that the control input $u(\cdot)$ is a bounded function of time (in other words, Dirac measures are excluded from the control action). This will be settled next in Theorem 1. Anyway, as will be explained in Section 2.2, this paper describes an intermediate step in the controllability study, which does not concern directly $u(\cdot)$. Let us give some important definitions and notations:

We denote A^i the ith row of the matrix A, B^i the ith entry of B, and let $I(q)$ the set of *active constraints*, i.e. $A^i q + B^i = 0$ for all $i \in I(q)$. The tangent cone to the domain $D = \{q : Aq + B \geqslant 0\}$ is defined as $V(q(t)) = \{x \in \mathbb{R}^{n_1/2} \times \mathbb{R}^{n_2/2} | A^i x = A_1^i x_1 + A_2^i x_2 \geqslant 0, i \in I(q)\}$, $V(q(t)) = \mathbb{R}^{n_1/2 + n_2/2}$ if $Aq + B > 0$, and $V(q(t)) = \emptyset$ if $Aq + B < 0$. Let us choose Moreau's collision rule with restitution e. We define $V_i(q(t)) = \{x \in \mathbb{R}^{n_1/2} \times \mathbb{R}^{n_2/2} | A^i x = A_1^i x_1 + A_2^i x_2 \geqslant 0\}$, $1 \leqslant i \leqslant m$, and let $M = \mathrm{blockdiag}(M_1, M_2)$. We will also use $[\cdot]_+$ to denote the positive part of $[\cdot]$ and $[\cdot]_-$ the negative part (i.e. $[\cdot]_+ = [\cdot]$ if $[\cdot] \geqslant 0$ and $[\cdot]_+ = 0$ if $[\cdot] < 0$, and similarly for negative part). Note that $\mathrm{prox}_M(y, V_i(q(t_k))) = y - \langle y, n_i \rangle_+ n_i$ with

$$n_i = \frac{-1}{\sqrt{A^i M^{-1} (A^i)^{\mathrm{T}}}} \begin{pmatrix} M_1^{-1}(A_1^i)^{\mathrm{T}} \\ M_2^{-1}(A_2^i)^{\mathrm{T}} \end{pmatrix} = \frac{-1}{\sqrt{A^i M^{-1} (A^i)^{\mathrm{T}}}} \begin{pmatrix} N_1^i \\ N_2^i \end{pmatrix}, \quad i \in \{1, \ldots, m\}$$

and the scalar product $\langle \cdot, \cdot \rangle$ is in the kinetic metric. Consequently, we can rewrite the impact law with restitution $e \in [0, 1]$ as [17]

$$\dot{q}(t_k^+) = -e\dot{q}(t_k^-) + (1+e)\left[\dot{q}(t_k^-) - \frac{1}{A^i M^{-1}(A^i)^{\mathrm{T}}}[A_1^i \dot{q}_1(t_k^-) + A_2^i \dot{q}_2(t_k^-)]_- \begin{pmatrix} N_1^i \\ N_2^i \end{pmatrix}\right]. \tag{4}$$

The choice of the model, especially the impact rule, will not be discussed here. Note that there could be a different e_i for each constraint. For the sake of simplicity of the subsequent presentation, we assume that $e_i = e$, $1 \leqslant i \leqslant m$.

2.2. Well-posedness

The dynamics in (2) (or with the impact law in (4)) is a measure differential inclusion [22]. Introducing a function $v(\cdot)$ of local bounded variation that equals the velocity $\dot{q}(\cdot)$ Lebesgue almost everywhere, the so-called differential measure $\mathrm{d}v$ is introduced and the dynamics in (2) is rewritten as an equality of measures as

$$-M\,\mathrm{d}v + \begin{pmatrix} 0 \\ E \end{pmatrix} u(q(t), v(t^+), t)\,\mathrm{d}t \in N_{V(q(t))}(v(t^+)), \tag{5}$$

where $N_{V(q(t))}(v(t^+))$ is the normal cone to the tangent cone $V(q(t))$ at $v(t^+)$.[1] The next theorem is a compilation of Proposition 32, Problem $\mathscr{P}$, Theorems 8, 10 and Corollary 9 of [2], adapted to (2).

Theorem 1. *Assume that the row vectors A^i, $i \in I(q)$ of the matrix A are independent for all $q \in D$, and that $\|Eu(q, v, t)\|_q \leqslant l(t)(1+d(q, q(0))+\|v\|_q)$ with $l(t) \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^+)$, $d(q, q(0))$ is the Riemannian distance between q and the initial position $q(0)$, and $\|\cdot\|_q$ is the kinetic norm. Then solutions of* (5) *exist on $\mathbb{R}^+$ and are unique with $q(\cdot)$ absolutely continuous, whereas the velocity $v(\cdot)$ is right continuous of local bounded variation. Moreover, the acceleration is a measure $\mathrm{d}v = \{\ddot{q}\}\,\mathrm{d}t + \mathrm{d}\mu_a$, where $\{\ddot{q}\}$ is a Lebesgue integrable function, and $\mathrm{d}\mu_a$ is an atomic measure with a countable set of atoms on any compact time interval (atoms coincide with impact times).*

Theorem 1 therefore guarantees that provided the control $u(\cdot)$ satisfies the above constraint, then the closed-loop system is well-posed. Note, however, that solutions may in general be discontinuous with respect to initial conditions [2]. Also, left accumulations of velocity jumps may exist (like in the bouncing ball system with $e \in (0, 1)$) but right accumulations are excluded. In the sequel, we shall deal with controllability problems involving a finite number of velocity jumps. From a general point of view, we propose to study the controllability of jugglers as follows:

(1) Assume that a controller u exists that can drive the “robot” at arbitrary positions on the “object”’s orbit and with arbitrary velocity. Then study the existence of a path in the $(q, \dot{q})$-space that consists of a finite sequence $\mathscr{S}$ of positions and velocities at impact times t_k.
(2) Construct a trajectory $q_d(\cdot) \in \mathscr{C}^1[(t_k, t_{k+1})]$ which satisfies $(q_d(t_k), \dot{q}_d(t_k^+)) \in \mathscr{S}$, $Aq_d(t) + B \geqslant 0$ for all $t \in (t_k, t_{k+1})$.
(3) Find a control $u(\cdot)$ which satisfies the inclusion

$$\begin{pmatrix} 0 \\ E \end{pmatrix} u(t) \in \{M\ddot{q}_d(t)\} + N_{V(q_d(t))}(\dot{q}_d(t)) \tag{6}$$

for all $t \in (t_k, t_{k+1})$.

This way of studying the controllability of jugglers is quite a natural one, as the z_1-dynamics is controllable only through the Lagrange multiplier λ, i.e. when there is at least one constraint $h_i(z_1, z_2) \geqslant 0$ that is active ($h_i(z_1, z_2) = 0$). This corresponds to either pushing the “object” with the “robot” (phases of permanent contact), or hitting the “object” with the “robot”. We are concerned with the second type of task in this paper. Moreover, we deal only with step (1). Preliminary solutions for solving steps (2) and (3) have been indicated in [8,15]. It is noteworthy that the object’s trajectory $q_{1d}(\cdot)$ is given solely by the dynamics (ballistic constraints), as the object is free between impacts. Hence, (6) is equivalent to $Eu(t) \in \{M_2\ddot{q}_{2d}(t)\} + (0_{n_1/2} \;\; I_{n_2/2})N_{V(q_d(t))}(\dot{q}_d(t))$, where $0_{n_1/2} \in \mathbb{R}^{n_1/2 \times n_1/2}$ is the null matrix, and $I_{n_2/2}$ is the $\frac{n_2}{2} \times \frac{n_2}{2}$ identity matrix. The normal cone in the right-hand side of (6) is not equal to $\{0\}$ on (t_k, t_{k+1}) as positions are continuous and for all t_k there exists at least one i such that $A^i q_d(t_k) + B^i = 0$. A fourth step is stabilization of trajectories.

Despite what one might think, step (1) is a difficult one and this is why we focus on it in this paper.

3. A controllability criterion

3.1. Controllability through the impacts

As mentioned above, we chose in this study to control the object’s dynamics through impacts with the robot’s dynamics. Therefore, phases of persistent contact between these two parts of the system, are excluded from the following developments. Similarly, as in [8] we make the following.

[1] Since the function $h(z_1, z_2)$ in (1) is linear, these cones are the usual tangent and normal cones to a convex set [25].

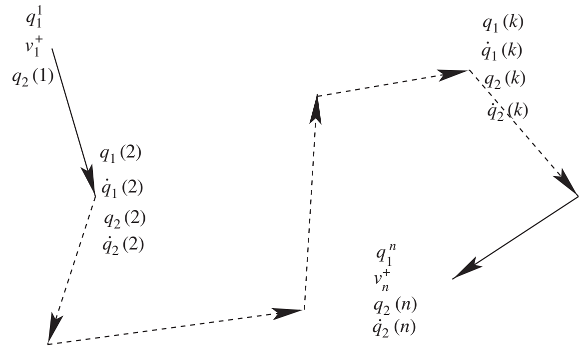

Fig. 3. The path from initial to final states.

Assumption 1. There exists $u(q, \dot{q}, t)$ satisfying the constraint in Theorem 1, such that $\dot{q}_2(k)$ can be given arbitrary values at arbitrary positions on the object's orbits. Moreover, it is possible to strike a unique arbitrary constraint at times t_k.

In this paper, *we deal only with simple impacts*, i.e. impacts with one constraint $A^i q + B^i = 0$ for some $1 \leqslant i \leqslant m$. Thus, we have by Theorem 1 the existence and uniqueness of a global solution. However, multiple impacts can be incorporated in the proposed framework by suitably modifying the reinitialization rule in (4) [22,4]. Assumption 1 allows one to decouple the control problem into two parts: the control of the object's dynamics using λ as the input (i.e. using impacts), then the control of the overall system with u. Here, we focus on the first part only and we suppose, throughout the paper, that Assumption 1 holds. Let us note that the algebraic form (3) of the dynamics at the impact times, allows one to express λ as a function of positions and velocities. Therefore, "using λ as the input" can be understood as using some of these variables as the input, in particular $\dot{q}_2(k)$ and $q_2(k)$ (which obviously is linked to $q_1(k)$, see (2)). Let us formulate the following controllability problem.

Problem 1. Given (q_1^1, v_1^+) and (q_1^n, v_n^+) with $v_1^+ \neq 0$, find a path $\{q_1(k)\}_{2 \leqslant k \leqslant n}$, $\{\dot{q}_1(k)\}_{2 \leqslant k \leqslant n}$, $\{q_2(k)\}_{1 \leqslant k \leqslant n}$ and $\{\dot{q}_2(k)\}_{2 \leqslant k \leqslant n}$, such that $(q_1(t_1), \dot{q}_1(t_1^+)) = (q_1^1, v_1^+)$ and $(q_1(t_n), \dot{q}_1(t_n^+)) = (q_1^n, v_n^+)$ (Fig. 3).

Note that $n \geqslant 2$ by construction. In Problem 1, n may be given or be considered as a variable (i.e. one may want to study the controllability with a fixed number of impacts, or search if there exists a finite n such that controllability holds). The times t_k are not fixed at this stage of the study. The constraint on v_1^+ will be made clear later. However, $v_n^+ = 0$ is a possible choice. Problem 1 is formulated such that an impact has occurred at t_1 and one starts looking at the system just after this impact. It will appear below (see (7)) that $\dot{q}_2(1)$ is not an unknown of the problem since we consider (q_1^1, v_1^+) and the time t_1 as initial data. This has no consequence on the rest of the study and has to be seen as an analysis artefact. One sees that if Problem 1 has a solution, then step (3) of the controllability study is to find out a controller $u(\cdot)$ which drives the robot state towards the values given by the sequences $\{q_2(k)\}_{1 \leqslant k \leqslant n}$ and $\{\dot{q}_2(k)\}_{2 \leqslant k \leqslant n}$.

Since our main goal is to characterize controllability, we will focus later on the characterization of reachable subspaces and the accessibility of the systems in (2) and (4).

Definition 1. For $n \geqslant 2$ the reachable set from (q_1^1, v_1^+) in $n-1$ impacts is the subset of $\mathbb{R}^{n_1/2} \times \mathbb{R}^{n_1/2}$ defined as $\mathscr{R}^n[(q_1^1, v_1^+)] = \{(q_1^n, v_n^+) |$ Problem 1 possesses at least one solution$\}$.

Definition 2. Let us denote $\bar{\mathscr{R}}^n[(q_1^1, v_1^+)] = \bigcup_{2 \leqslant k \leqslant n} \mathscr{R}^k[(q_1^1, v_1^+)]$. The object's dynamics is called accessible when $\bar{\mathscr{R}}^n[(q_1^1, v_1^+)]$ contains an open set for some $n \geqslant 2$, and accessible in $N-1$ impacts if $\bar{\mathscr{R}}^n[(q_1^1, v_1^+)]$ contains an open set for all $n \geqslant N$.

The reason for the $N-1$ instead of N comes from the fact that the first impact that counts in the analysis is at time t_2. Let us denote $\mathscr{R}_{V_1}^n[(q_1^1, v_1^+)]$ the reachable set from (q_1^1, v_1^+) in $n-1$ impacts, with object's trajectories $(q_1(k), \dot{q}_1(k))_{2 \leqslant k \leqslant n}$ remaining in a neighborhood V_1 of (q_1^1, v_1^+).

Definition 3. The object's dynamics is called locally accessible if $\bar{\mathscr{R}}_{V_1}^n[(q_1^1, v_1^+)]$ contains an open set for any V_1 and any $n \geqslant 2$. When the object's dynamics is accessible but not locally accessible, it is said to be globally accessible.

As pointed out in the Introduction, some jugglers are not locally accessible. In other words, given (q_1^1, v_1^+) and (q_1^n, v_n^+) in any neighborhood V_1, one may need to consider object's trajectories which do not remain inside V_1 to reach (q_1^n, v_n^+). This is essentially due to the unilaterality of the constraints (i.e. the complementarity conditions). Note that the trajectories are understood as the positions and velocities at the impact times only. However, the complete trajectories (i.e. the functions of time t) may leave V_1 as the simplest one degree-of-freedom juggler under gravity (see [8]) shows.

Definition 4 (*CTI*). The object's dynamics in (2) is controllable in $n-1$ impacts (or controllable through the impacts in $n-1$ impacts, denoted as CTI($n-1$)) if Problem 1 has a solution for all $(q_1^1, v_1^+) \in \mathbb{R}^{n_1/2} \times \mathbb{R}^{n_1/2}$ and all $(q_1^n, v_n^+) \in \mathbb{R}^{n_1/2 \times n_1/2}$.

Let us denote $x_1^{\mathrm{T}} = (q_1^{\mathrm{T}}(2), \dots, q_1^{\mathrm{T}}(n-1))$, $x_1 \in \mathbb{R}^{(n-2)(n_1/2)\times 1}$, $x_2^{\mathrm{T}} = (q_2^{\mathrm{T}}(1), \dots, q_2^{\mathrm{T}}(n))$, $x_2 \in \mathbb{R}^{n(n_2/2)\times 1}$, $x_3^{\mathrm{T}} = (\dot{q}_1^{\mathrm{T}}(2), \dots, \dot{q}_1^{\mathrm{T}}(n-1))$, $x_3 \in \mathbb{R}^{(n-2)(n_1/2)\times 1}$, $x_4^{\mathrm{T}} = (\dot{q}_2^{\mathrm{T}}(2), \dots, \dot{q}_2^{\mathrm{T}}(n))$, $x_4 \in \mathbb{R}^{(n-1)(n_2/2)\times 1}$. The first aim of this note is to prove the following.

Lemma 1. *Problem* 1 *has a solution if and only if the constrained equation* $H_j(x_1, x_2, x_4)=0$, $G_j(x_1, x_2, x_4) \geqslant 0$ *has a solution*, for at least one $j \in \{1, \dots, n^{(n-2)}\}$ *where* $H_j(\cdot)$ *and* $G_j(\cdot)$ *are the nonlinear functions given in* (36).

This result is not surprising due to the complementarity conditions in the dynamics. It is the basis for subsequent analysis.

Proof. The proof (i.e. the construction of the functions $H_j(\cdot)$ and $G_j(\cdot)$) is divided in five steps which correspond to the constraints that the unknowns in Problem 1 have to satisfy:

Final velocity equality: In the following, the index $i(k)$ means that the constraint $i(k)$, corresponding to the row $i(k)$ of the matrix A, is striked at the time t_k. For instance, if $m=2$ one may have $i(k)=1$ or $i(k)=2$. Since Problem 1 is concerned with $n-1$ impacts, let us denote $\mathscr{I}_j = \{i(2), \dots, i(n)\}$, $j \in \{1, \dots, m^{(n-1)}\}$, the possible sequences of successive simple impacts with the m constraints. From (4) we have

$$\dot{q}_1(t_k^+) = \dot{q}_1(k) - (1+e)\frac{1}{A^{i(k)}M^{-1}(A^{i(k)})^{\mathrm{T}}}[A_1^{i(k)}\dot{q}_1(k) + A_2^{i(k)}\dot{q}_2(k)]_- N_1^{i(k)} = \dot{q}_1(k) - a_-^k N_1^{i(k)}, \tag{7}$$

with $a_-^k = \frac{1+e}{A^{i(k)}M^{-1}(A^{i(k)})^{\mathrm{T}}}[A_1^{i(k)}\dot{q}_1(k) + A_2^{i(k)}\dot{q}_2(k)]_-$ and $\dot{q}_1(k) \triangleq \dot{q}_1(t_k^-)$. One notes that a_-^k is linear in its arguments $\dot{q}_1(k)$ and $\dot{q}_2(k)$ provided $A_1^{i(k)}\dot{q}_1(k) + A_2^{i(k)}\dot{q}_2(k) \leqslant 0$. From (2) it follows that

$$\dot{q}_1(k) = \dot{q}_1(t_{k-1}^+). \tag{8}$$

From (7) and (8) one can write

$$\begin{cases} \dot{q}_1(t_k^+) = \dot{q}_1(t_{k-1}^+) - a_-^k N_1^{i(k)}, \\ \dot{q}_1(t_{k-1}^+) = \dot{q}_1(t_{k-2}^+) - a_-^{k-1} N_1^{i(k-1)}, \\ \vdots \\ \dot{q}_1(t_2^+) = \dot{q}_1(t_1^+) - a_-^2 N_1^{i(2)}. \end{cases} \tag{9}$$

Consequently there are $m^{(n-1)}$ possible sequences as (9) within the formulation of Problem 1. We obtain

$$\dot{q}_1(t_k^+) = \dot{q}_1(t_1^+) - \left(\sum_{j=2}^{k} a_-^j N_1^{i(j)}\right). \tag{10}$$

From Problem 1 we can rewrite (10) as

$$\dot{q}_1(t_k^+) = v_1^+ - \left(\sum_{j=2}^{k} a_-^j\,(\dot{q}_1(j), \dot{q}_2(j))\,N_1^{i(j)}\right) \tag{11}$$

and obviously at t_n one obtains

$$v_n^+ = v_1^+ - \left(\sum_{j=2}^{n} a_-^j(\dot{q}_1(j), \dot{q}_2(j))N_1^{i(j)}\right), \tag{12}$$

where v_n^+ and v_1^+ are data of the problem. Consider (7) and (8), and the linearity of a_-^j; in particular $a_-^1 = \frac{1+e}{A^{i(1)}M^{-1}(A^{i(1)})^{\mathrm{T}}}[A_1^{i(1)}v_1^+ + A_2^{i(1)}\dot{q}_2(1)]_-$ and using (7) and (9) one can express $\dot{q}_1(j)$ as a linear function of x_4. For instance, one has

$$\dot{q}_1(j) = \dot{q}_1(j-1) - a_-^{j-1}(\dot{q}_1(j-2) - a_-^{j-2}N_1^{i(j-2)}, \dot{q}_2(j-1))N_1^{i(j-1)}. \tag{13}$$

We have $\dot{q}_1(2) = v_1^+$, $\dot{q}_1(3) = v_1^+ - a_-^2(v_1^+, \dot{q}_2(2))N_1^{i(2)}$, $\dot{q}_1(4) = v_1^+ - a_-^3[v_1^+ - a_-^2(v_1^+, \dot{q}_2(2))N_1^{i(2)}, \dot{q}_2(3)]N_1^{i(3)} - a_-^2(v_1^+, \dot{q}_2(2))N_1^{i(2)}$, and so on. Then from (11) and since $A_1^{i(k)}\dot{q}_1(k) + A_2^{i(k)}\dot{q}_2(k)$ is a scalar, one gets

$$\dot{q}_1(t_k^+) = v_1^+ + \mathscr{F}_k(e, A, M_1, M_2)x_4 + \mathscr{G}_k(v_1^+, e, M_1, A, M_2), \tag{14}$$

where $\mathscr{F}_k(\cdot)$ and $\mathscr{G}_k(\cdot)$ are constant matrices. Taking $k = n$ one notes that the equality in (14) represents a constraint on x_4 given as

$$\mathscr{F}_{\mathscr{I}_j}(e, A, M_1, M_2)x_4 + \mathscr{G}_{\mathscr{I}_j}(v_1^+, e, M_1, A, M_2) + v_n^+ = 0, \tag{15}$$

where $\mathscr{F}_{\mathscr{I}_j} \in \mathbb{R}^{n_1/2\times(n_2/2)(n-1)}$ and $\mathscr{G}_{\mathscr{I}_j} \in \mathbb{R}^{n_1/2\times 1}$.

Example 3. Let us illustrate the fashion to obtain (15) for Example 2. We choose $n = 3$, $i(2) = 2$, $i(3) = 1$ and denote the sequence as $\mathscr{I}_1$. To simplify the notations we take $m_1 = m_2 = m_3 = 1$. One obtains

$$\dot{q}_1(t_2^+) = v_1^+ + \frac{1+e}{2}[\cos\theta_2\dot{x}(2) + \sin\theta_2\dot{y}(2) - \dot{\zeta}_2(2)]\begin{pmatrix}\cos\theta_2\\ \sin\theta_2\end{pmatrix} \tag{16}$$

and

$$\dot{q}_1(t_3^+) = v_3^+ = \dot{q}_1(t_2^+) + \frac{1+e}{2}[\cos\theta_1\dot{x}(3) + \sin\theta_1\dot{y}(3) - \dot{\zeta}_1(3)]\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix}. \tag{17}$$

Inserting equality (16) into (17) one gets an illustration of (12) for the nonprehensile manipulation system. We have $x_4^{\mathrm{T}} = (\dot{\zeta}_1(2)\dot{\zeta}_2(2)\dot{\zeta}_1(3)\dot{\zeta}_2(3))$. Using (8) for $k = 2$ and $k = 3$, and using (14), (16) and (17), we obtain

$$\mathscr{F}_2 = \begin{pmatrix} 0 & \dfrac{1+e}{2}\cos\theta_2 & 0 & 0\\ 0 & \dfrac{1+e}{2}\sin\theta_2 & 0 & 0\end{pmatrix}. \tag{18}$$

and

$$\mathscr{F}_{\mathscr{I}_1}(e, A, M_1, M_2) = \begin{pmatrix} 0 & \dfrac{1+e}{2}\cos\theta_2 - \dfrac{(1+e)^2}{4}(\cos^2\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\sin\theta_1) & \dfrac{1+e}{2}\cos\theta_1 & 0\\ 0 & \dfrac{1+e}{2}\sin\theta_2 - \dfrac{(1+e)^2}{4}(\sin^2\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2\cos\theta_1) & \dfrac{1+e}{2}\sin\theta_1 & 0\end{pmatrix}. \tag{19}$$

The constant matrices $\mathscr{G}_2$ in (14) and $\mathscr{G}_{\mathscr{I}_1}$ can be calculated similarly.

Negative pre-impact velocities: One has from the impact existence condition $A_1^{i(k)}\dot{q}_1(k) + A_2^{i(k)}\dot{q}_2(k) \leqslant 0$ and (8)

$$A_1^{i(k+1)}\dot{q}_1(t_k^+) + A_2^{i(k+1)}\dot{q}_2(k+1) \leqslant 0,\ 1 \leqslant k \leqslant n-1. \tag{20}$$

Still using recursively (13) it follows that (20) can be rewritten compactly as

$$\mathscr{K}_{\mathscr{I}_j}(v_1^+, A, e, M_1, M_2) + \mathscr{L}_{\mathscr{I}_j}(e, M_1, A, M_2)x_4 \geqslant 0 \tag{21}$$

for some constant matrices $\mathscr{K}_{\mathscr{I}_j}(\cdot) \in \mathbb{R}^{(n-1)\times 1}$, $\mathscr{L}_{\mathscr{I}_j}(\cdot) \in \mathbb{R}^{(n-1)\times(n_2/2)(n-1)}$ and the $\geqslant 0$ is componentwise.

Nonsingularity conditions: To be able to go further in our analysis (see below), we are led to take from now on the following assumption:

$$A_1^{i(k+1)}\dot{q}_1(t_k^+) \neq 0, \quad 0 \leqslant k \leqslant n-1. \tag{22}$$

This gives us, using (14), an additional constraint on x_4:

$$A_1^{i(k+1)}v_1^+ + A_1^{i(k+1)}\mathscr{F}_k x_4 + A_1^{i(k+1)}\mathscr{G}_k \neq 0, \quad 1 \leqslant k \leqslant n-1. \tag{23}$$

The matrices $\mathscr{F}_k$ and $\mathscr{G}_k$ are constant matrices at each step k, see an instance of calculation in Example 3. Concatenating these $n-1$ inequalities one obtains

$$\mathscr{U}_{\mathscr{I}_j}(A)v_1^+ + \mathscr{V}_{\mathscr{I}_j}(e, A, M_1, M_2)x_4 + \mathscr{W}_{\mathscr{I}_j}(e, A, M_1, M_2, v_1^+) > 0 \quad \text{or} \quad < 0, \tag{24}$$

where $\mathscr{U}_{\mathscr{I}_j} \in \mathbb{R}^{(n-2)\times n_1/2}$, $\mathscr{V}_{\mathscr{I}_j} \in \mathbb{R}^{(n-2)\times(n-1)(n_2/2)}$, and $\mathscr{W}_{\mathscr{I}_j} \in \mathbb{R}^{(n-2)\times 1}$. Note that if $n=2$ then $\mathscr{V}_{\mathscr{I}_j}(e, A, M_1, M_2)=0$ and $\mathscr{W}_{\mathscr{I}_j}=0$ since $\mathscr{F}_2=0$ and $\mathscr{G}_2=0$ (see (14)).

Final position equality: From the object's dynamics one gets

$$q_1(k+1) - q_1(k) = \dot{q}_1(t_k^+)\Delta_k, \tag{25}$$

with $\Delta_k = t_{k+1} - t_k$, $1 \leqslant k \leqslant n-1$, and from the constraints expression at t_{k+1}

$$A_1^{i(k+1)} q_1(k+1) + A_2^{i(k+1)} q_2(k+1) + B^{i(k+1)} = 0, \quad 0 \leqslant k \leqslant n-1 \tag{26}$$

and inserting (25)

$$A_1^{i(k+1)} q_1(k) + A_2^{i(k+1)} q_2(k+1) + B^{i(k+1)} + A_1^{i(k+1)}\dot{q}_1(t_k^+)\Delta_k = 0. \tag{27}$$

Now, from (25) it easily follows that

$$q_1^n - q_1^1 = \sum_{k=1}^{n-1} \Delta_k \dot{q}_1(t_k^+). \tag{28}$$

Since we have assumed that $A_1^{i(k+1)}\dot{q}_1(t_k^+) \neq 0$, $0 \leqslant k \leqslant n-1$, we can use (27) to express Δ_k. Inserting into (28) and using (26), we obtain

$$\begin{aligned} q_1^n - q_1^1 &= \sum_{k=1}^{n-1} \frac{-\dot{q}_1(t_k^+)}{A_1^{i(k+1)}\dot{q}_1(t_k^+)} [A_1^{i(k+1)} q_1(k) + A_2^{i(k+1)} q_2(k+1) + B^{i(k+1)}] \\ &= \sum_{k=1}^{n-1} \frac{-\dot{q}_1(t_k^+)}{A_1^{i(k+1)}\dot{q}_1(t_k^+)} A_1^{i(k+1)} [q_1(k) - q_1(k+1)], \end{aligned} \tag{29}$$

where q_1^n and q_1^1 are data of the problem. Now, using the fact that we can express $\dot{q}_1(t_k^+)$ as a function of v_1^+ and x_4, see (14), for all $k \in \{2, \ldots, n\}$, we obtain from the second equality in (29)

$$\mathscr{A}_{\mathscr{I}_j}(v_1^+, A, M_1, M_2, x_4)x_1 + \mathscr{B}_{\mathscr{I}_j}(v_1^+, A, q_1^1) + \mathscr{R}_{\mathscr{I}_j}(e, A, M_1, M_2, v_1^+, x_4)q_1^n = 0 \tag{30}$$

for some matrices $\mathscr{A}_{\mathscr{I}_j}(\cdot) \in \mathbb{R}^{n_1/2\times n_1/2(n-2)}$, $\mathscr{B}_{\mathscr{I}_j}(\cdot) = -q_1^1 + \frac{v_1^+}{A_1^{i(2)}v_1^+} A_1^{i(2)} q_1^1 \in \mathbb{R}^{n_1/2\times 1}$ and $\mathscr{R}_{\mathscr{I}_j}(\cdot) = I_{n_1/2} - \frac{\dot{q}_1(t_{n-1}^+)A_1^{i(n)}}{A_1^{i(n)}\dot{q}_1(t_{n-1}^+)} \in \mathbb{R}^{n_1/2\times n_1/2}$ whose entries may be singular in x_4, but with a denominator that is linear in x_4, see (14).

Example 4. Let us continue with the nonprehensile manipulation system as in Examples 2 and 3. We have $x_1^{\mathrm{T}} = (x(2)\ \ y(2))$. Using (29) with $n=3$, the same sequence $\mathscr{I}_1$ as in Example 3, we get

$$\begin{aligned} &\mathscr{A}_{\mathscr{I}_1}(v_1^+, A, x_4) \\ &= \begin{pmatrix} \dfrac{\dot{x}_1^+ \cos\theta_2}{\cos\theta_2 \dot{x}_1^+ + \sin\theta_2 \dot{y}_1^+} - \dfrac{\dot{x}(t_2^+)\cos\theta_1}{\cos\theta_1 \dot{x}(t_2^+) + \sin\theta_1 \dot{y}(t_2^+)} & \dfrac{\dot{x}_1^+ \sin\theta_2}{\cos\theta_2 \dot{x}_1^+ + \sin\theta_2 \dot{y}_1^+} - \dfrac{\dot{x}(t_2^+)\sin\theta_1}{\cos\theta_1 \dot{x}(t_2^+) + \sin\theta_1 \dot{y}(t_2^+)} \\ \dfrac{\dot{y}_1^+ \cos\theta_2}{\cos\theta_2 \dot{x}_1^+ + \sin\theta_2 \dot{y}_1^+} - \dfrac{\dot{y}(t_2^+)\cos\theta_1}{\cos\theta_1 \dot{x}(t_2^+) + \sin\theta_1 \dot{y}(t_2^+)} & \dfrac{\dot{y}_1^+ \sin\theta_2}{\cos\theta_2 \dot{x}_1^+ + \sin\theta_2 \dot{y}_1^+} - \dfrac{\dot{y}(t_2^+)\sin\theta_1}{\cos\theta_1 \dot{x}(t_2^+) + \sin\theta_1 \dot{y}(t_2^+)} \end{pmatrix}, \end{aligned} \tag{31}$$

where $v_1^+ = (\dot{x}_1^+\ \ \dot{y}_1^+)^{\mathrm{T}}$. Using now (16) to calculate $\dot{x}(t_2^+)$ and $\dot{y}(t_2^+)$ allows one to get the final form of $\mathscr{A}_{\mathscr{I}_1}(v_1^+, A, x_4)$, obtained from the second equality in (29). Since the denominators in (31) are linear with respect to $\dot{x}(t_2^+)$ and $\dot{y}(t_2^+)$, and since from (16) both these quantities are linear in x_4, it follows that the denominators are linear in x_4 as well. The other matrices in (30) are calculated similarly.

Positive flight-times: From (25) and (27) we get

$$A_1^{i(k+1)}[q_1(k+1) - q_1(k)] = A_1^{i(k+1)}\dot{q}_1(t_k^+)\Delta_k. \tag{32}$$

Since $A_1^{i(k+1)}\dot{q}_1(t_k^+) \neq 0$, the constraints $\Delta_k \geqslant 0$, $1 \leqslant k \leqslant n-1$, may therefore be written as (see (14))

$$\mathscr{H}_{\mathscr{I}_j}(v_1^+, e, A, M_1, M_2, x_4)x_1 + \mathscr{J}_{\mathscr{I}_j}(q_1^1, q_1^n, v_1^+, e, A, M_1, M_2, x_4) \geqslant 0 \tag{33}$$

for some matrices $\mathscr{H}_{\mathscr{I}_j}(\cdot) \in \mathbb{R}^{(n-1)\times(n_1/2)(n-2)}$ and $\mathscr{J}_{\mathscr{I}_j}(\cdot) \in \mathbb{R}^{(n-1)\times 1}$ whose components may be singular in x_4. The vector $\mathscr{J}_{\mathscr{I}_j}$ has the following structure

$$\begin{pmatrix} -\frac{A_1^{i(2)}q_1^1}{A_1^{i(2)}v_1^+} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{A_1^{i(n)}q_1^n}{A_1^{i(n)}\left[v_1^+ - \sum_{j=2}^{n-1} a_-^j N_1^{i(j)}\right]} \end{pmatrix}. \tag{34}$$

Note that the form of the inequalities in (33) depends on the sign of $A_1^{i(k+1)}\dot{q}_1(t_k^+)$ for each $1 \leqslant k \leqslant n-1$. From (34) one sees that $\mathscr{J}_{\mathscr{I}_j}$ is linear in q_1^1 and in q_1^n.

Unilateral constraints active at impact times: From (26) one can construct the set of equalities

$$\mathscr{C}_{\mathscr{I}_j}(A)x_1 + \mathscr{D}_{\mathscr{I}_j}(A)x_2 + \mathscr{E}_{\mathscr{I}_j}(B, q_1^1) + \mathscr{Q}_{\mathscr{I}_j}(A)q_1^n = 0, \tag{35}$$

with $\mathscr{C}_{\mathscr{I}_j}(\cdot) \in \mathbb{R}^{n\times(n_1/2)(n-2)}$, $\mathscr{D}_{\mathscr{I}_j}(\cdot) \in \mathbb{R}^{n\times n(n_2/2)}$, $\mathscr{E}_{\mathscr{I}_j}(\cdot) \in \mathbb{R}^{n\times 1}$, and

$$\mathscr{Q}_{\mathscr{I}_j}(A)q_1^n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_1^{i(n)}q_1^n \end{pmatrix}, \quad \mathscr{Q}_{\mathscr{I}_j} \in \mathbb{R}^{n\times n_1/2}.$$

The functions $H_j(\cdot)$, $G_j(\cdot)$ can be constructed from (30), (35), (15) and (33), (21), (24), respectively. The controllability problem formulated as Problem 1 yields nonlinear equations[2] in x under inequality constraints, as follows:

$$(S_1)\quad \begin{cases} \mathscr{A}_{\mathscr{I}_j}(v_1^+, A, M_1, M_2, x_4)x_1 + \mathscr{B}_{\mathscr{I}_j}(v_1^+, A, q_1^1) + \mathscr{R}_{\mathscr{I}_j}(e, A, M_1, M_2, v_1^+, x_4)q_1^n = 0 & \text{(i)} \\ \mathscr{C}_{\mathscr{I}_j}(A)x_1 + \mathscr{D}_{\mathscr{I}_j}(A)x_2 + \mathscr{E}_{\mathscr{I}_j}(B) + \mathscr{Q}_{\mathscr{I}_j}(A)q_1^n = 0 & \text{(ii)} \\ \mathscr{F}_{\mathscr{I}_j}(e, A, M_1, M_2)x_4 + \mathscr{G}_{\mathscr{I}_j}(v_1^+, e, M_1, M_2, A) + v_n^+ = 0 & \text{(iii)} \\ \mathscr{H}_{\mathscr{I}_j}(v_1^+, e, A, M_1, M_2, x_4)x_1 + \mathscr{J}_{\mathscr{I}_j}(q_1^1, q_1^n, v_1^+, e, A, M_1, M_2, x_4) \geqslant 0 & \text{(iv)} \\ \mathscr{K}_{\mathscr{I}_j}(v_1^+, A, e, M_1, M_2) + \mathscr{L}_{\mathscr{I}_j}(e, M_1, M_2, A)x_4 \geqslant 0 & \text{(v)} \\ \mathscr{U}_{\mathscr{I}_j}(A)v_1^+ + \mathscr{V}_{\mathscr{I}_j}(e, A, M_1, M_2)x_4 + \mathscr{W}_{\mathscr{I}_j}(e, A, M_1, M_2, v_1^+) > 0 \text{ or } < 0 & \text{(vi)} \\ j \in \{1, \ldots, m^{(n-1)}\} \end{cases} \tag{36}$$

which forms two sets of $n_1 + n$ equalities and $3n - 4$ inequalities for each j. We have therefore shown that Problem 1 has a solution only if the constrained equation (S_1) has a solution for at least one j. Since the converse is obviously true Lemma 1 is proved. □

Remark 1. To satisfy the nonsingularity (22), we can assume, for example, that the matrix A_1 satisfies $\mathrm{rank}(A_1) \geqslant 2$. Then Eq. (vi) below is automatically satisfied and should be removed, which simplifies substantially the analysis. However, this assumption is meaningful only if $m \geqslant 2$ and $n_1 \geqslant 4$. It will be convenient in deriving the accessibility criterion. Physically, it means that the object's velocity is never orthogonal to the chosen constraint to be striked. For instance, in Example 2,

$$A_1 = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ \cos\theta_2 & \sin\theta_2 \end{pmatrix}.$$

[2] It is interesting to note that the controllability problem as formulated in [16] also results in a nonlinear program with equality and inequality constraints.

So provided $\theta_2 \neq \theta_1$, this assumption is satisfied (take, for example, $0 < \theta_1 < \frac{\pi}{2}$, $\frac{\pi}{2} < \theta_2 < \pi$). The situation $\theta_2 = \theta_1$ corresponds to the two plates with parallel directions, which obviously is not a good design choice. This row rank assumption and the rank assumption in Theorem 1 are obviously decoupled one from each other.

Remark 2. It clearly appears from (25) and (27) that $q_2(k)$ cannot be used as an input in a mapping for the computation of $q_1(k)$.

Remark 3. The constraint that $v_1^+ \neq 0$ in Problem 1 appears as a necessary condition for the construction of the matrices. It is in fact an artefact due to the way we have formulated the controllability in Problem 1. A solution for removing this technical assumption is to strike the object initially and then continuing. However, the choice $v_n^+ = 0$ is possible, indeed the denominators in (29) do not involve v_n^+, but only $\dot{q}_1(t_n^-) = \dot{q}_1(t_{n-1}^+)$.

3.2. *Some properties of the constrained equation*

The set of equalities/inequalities in (36) has the following useful properties:*An equivalent formulation*: If we use the first equality in (29) we get

$$\mathscr{A}_{\mathscr{I}_j}(v_1^+, A, M_1, M_2, x_4)x_1 + \mathscr{B}_{\mathscr{I}_j}(v_1^+, A, M_1, M_2, x_4)x_2 + \mathscr{R}_{\mathscr{I}_j}(A, B, q_1^1) + q_1^n = 0 \quad \text{(i}'\text{)}, \tag{37}$$

where we use the same notations than in (30) but the matrices are different. Here, we have $\mathscr{R}_{\mathscr{I}_j}(A, B, q_1^1) = -q_1^1 + \frac{v_1^+}{A_1^{i(2)} v_1^+}[A_1^{i(2)} q_1^1 + B^{i(2)}]$. The first column of $\mathscr{B}_{\mathscr{I}_j}$ in (37) is made of zeros because $q_2(1)$ is not in the right-hand side of the first line of (29). Let us denote as (S_2) the constrained equation made of (37) and (ii)–(vi) in (36). The constrained equation (S_2) is equivalent to (S_1). Indeed, denoting the first equality of (29) as $(29)_1$ and the second equality as $(29)_2$, we obtain $[(29)_1$ and $(26)] \Rightarrow [(29)_2]$ (and (26)). Conversely, $[(29)_2$ and $(26)] \Rightarrow [(29)_1]$ (and (26)). Since (26) is imposed in both (S_1) and (S_2) (through (35) numbered (ii)), they are equivalent one to each other. The choice between (S_1) and (S_2) is difficult at this stage of the study and may depend on some additional constraints (e.g. numerical issues). As we shall see below, for one degree-of-freedom objects ($n_1 = 2$) then only the formalism (S_2) makes sense because the second line of (29) then becomes trivial and is not usable.

Example 5. Continuing Example 4 and using the first equality in (29), we obtain instead of (31) the following matrix $\mathscr{A}_{\mathscr{I}_1}$ for (S_2):

$$\mathscr{A}_{\mathscr{I}_1}(v_1^+, A, M_1, M_2, x_4) = \begin{pmatrix} \dfrac{-\dot{x}(t_2^+)\cos\theta_1}{\cos\theta_1 \dot{x}(t_2^+) + \sin\theta_1 \dot{y}(t_2^+)} & \dfrac{-\dot{x}(t_2^+)\sin\theta_1}{\cos\theta_1 \dot{x}(t_2^+) + \sin\theta_1 \dot{y}(t_2^+)} \\ \dfrac{-\dot{y}(t_2^+)\cos\theta_1}{\cos\theta_1 \dot{x}(t_2^+) + \sin\theta_1 \dot{y}(t_2^+)} & \dfrac{-\dot{y}(t_2^+)\sin\theta_1}{\cos\theta_1 \dot{x}(t_2^+) + \sin\theta_1 \dot{y}(t_2^+)} \end{pmatrix} \tag{38}$$

which can be put into its final form using (16) and (8).

Linearity in the final state: Let $x_n^{\mathrm{T}} = ((q_1^n)^{\mathrm{T}}, (v_n^+)^{\mathrm{T}})$. Then (i) (i′) (ii) (iii) and (iv) are linear in x_n, so that both (S_1) and (S_2) can be rewritten as

$$\begin{cases} \mathscr{M}_{\mathscr{I}_j}(x_4, v_1^+)x_n + \mathscr{N}_{\mathscr{I}_j}(x_1, x_2, x_4, q_1^1, v_1^+) = 0 & \text{(i) (or (i}'\text{)) (ii) (iii)}, \\ \mathscr{P}_{\mathscr{I}_j}(x_4, v_1^+)q_1^n + \mathscr{T}_{\mathscr{I}_j}(x_1, x_4, q_1^1, v_1^+) \geqslant 0 & \text{(iv)–(vi)}, \end{cases} \tag{39}$$

where

$$\mathscr{M}_{\mathscr{I}_j}(x_4, v_1^+) = \begin{pmatrix} \mathscr{R}_{\mathscr{I}_j}(x_4, v_1^+) & 0 \\ \mathscr{Q}_{\mathscr{I}_j} & 0 \\ 0 & I_{n_1/2} \end{pmatrix} \text{ for } (S_1), \quad \mathscr{M}_{\mathscr{I}_j} = \begin{pmatrix} I_{n_1/2} & 0 \\ \mathscr{Q}_{\mathscr{I}_j} & 0 \\ 0 & I_{n_1/2} \end{pmatrix} \text{ for } (S_2),$$

$$\mathscr{N}_{\mathscr{I}_j}(x_1, x_2, x_4, q_1^1, v_1^+) = \begin{pmatrix} \mathscr{A}_{\mathscr{I}_j}(x_4, v_1^+)x_1 + \mathscr{B}_{\mathscr{I}_j}(x_4, v_1^+, q_1^1) \\ \mathscr{C}_{\mathscr{I}_j}x_1 + \mathscr{D}_{\mathscr{I}_j}x_2 + \mathscr{E}_{\mathscr{I}_j} \\ \mathscr{F}_{\mathscr{I}_j}x_4 + \mathscr{G}_{\mathscr{I}_j}(v_1^+) \end{pmatrix} \text{ for } (S_1),$$

$$\mathscr{N}_{\mathscr{I}_j}(x_1, x_2, x_4, q_1^1, v_1^+) = \begin{pmatrix} \mathscr{A}_{\mathscr{I}_j}(x_4, v_1^+)x_1 + \mathscr{B}_{\mathscr{I}_j}(x_4, v_1^+) + \mathscr{R}_{\mathscr{I}_j}(x_4, q_1^1) \\ \mathscr{C}_{\mathscr{I}_j}x_1 + \mathscr{D}_{\mathscr{I}_j}x_2 + \mathscr{E}_{\mathscr{I}_j} \\ \mathscr{F}_{\mathscr{I}_j}x_4 + \mathscr{G}_{\mathscr{I}_j}(v_1^+) \end{pmatrix} \text{ for } (S_2),$$

$$\mathscr{P}_{\mathscr{I}_j}(x_4, v_1^+) = \begin{pmatrix} 0_{(n-2)\times n_1/2} \\ \dfrac{A_1^{i(n)}}{A_1^{i(n)}[v_1^+ - \sum_{j=2}^{n-1} a_-^j N_1^{i(j)}]} \end{pmatrix}, \quad \mathscr{T}_{\mathscr{I}_j}(x_1, x_4, q_1^1, v_1^+) = \mathscr{H}_{\mathscr{I}_j}(x_4, v_1^+)x_1 + \begin{pmatrix} -\dfrac{A_1^{i(2)}q_1^1}{A_1^{i(2)}v_1^+} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The two functions H_j and G_j in Lemma 1 satisfy $H_j(x_1, x_2, x_4) = H_{j,x_4}(x_1, x_2)$ and $G_j(x_1, x_2, x_4) = G_{j,x_4}(x_1, x_2)$, where $H_{j,x_4}(\cdot)$ and $G_{j,x_4}(\cdot)$ are linear. Since x_4 and x_2 play the role of the input, this means that the problem is linear in the state but nonlinear in the input.

Further equivalent formulations: (a) One can formulate Problem 1 and rewrite (36) as

$$\begin{cases} F_j(x_1, x_4) = 0 \quad \text{(i)(iii)}, \\ (x_1, x_4) \in \Omega_j \subset \mathbb{R}^{(n-2)[(n_1+n_2+2)/2]} \quad \text{(iv)(v)(vi)(ii)} \end{cases} \tag{40}$$

for all $j \in \{1, \ldots, m^{(n-1)}\}$, some function $F_j(\cdot)$ and some closed subset Ω_j. The problem is then under the canonical form studied in [21] for the variables x_1 and x_4. However, the applicability of the general conditions of existence of a solution to (40) in [21] is not straightforward.

(b) Let us denote $x^{\mathrm{T}} = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, x_4^{\mathrm{T}})$. Then (S_1) in (36) can be written as

$$\begin{cases} H_j(x_1, x_2, x_4) = M_j(x_4)x + N_j(x_4) = 0 \quad \text{(i)(ii)(iii)}, \\ G_j(x_1, x_2, x_4) = L_j(x_4)x + K_j(x_4) \geqslant 0 \quad \text{(iv)(v)(vi)}, \end{cases} \tag{41}$$

where

$$M_j(x_4) = \begin{pmatrix} \mathscr{A}_{\mathscr{I}_j}(x_4) & 0 & 0 \\ \mathscr{C}_{\mathscr{I}_j} & \mathscr{D}_{\mathscr{I}_j} & 0 \\ 0 & 0 & \mathscr{F}_{\mathscr{I}_j} \end{pmatrix} \quad \text{and} \quad N_j(x_4) = \begin{pmatrix} \mathscr{B}_{\mathscr{I}_j}(x_4) \\ \mathscr{E}_{\mathscr{I}_j} \\ \mathscr{G}_{\mathscr{I}_j} \end{pmatrix} + \mathscr{M}_{\mathscr{I}_j}(x_4)x_n,$$

$$L_j(x_4) = \begin{pmatrix} \mathscr{H}_{\mathscr{I}_j}(x_4) & 0 & 0 \\ 0 & 0 & \mathscr{L}_{\mathscr{I}_j} \end{pmatrix},$$

i.e. $\mathscr{N}_{\mathscr{I}_j}$ in (39) is

$$\mathscr{N}_{\mathscr{I}_j} = M_j(x_4)x + \begin{pmatrix} \mathscr{B}_{\mathscr{I}_j}(x_4) \\ \mathscr{E}_{\mathscr{I}_j} \\ \mathscr{G}_{\mathscr{I}_j} \end{pmatrix}.$$

In (41) $M_j \in \mathbb{R}^{(n_1+n)\times(n_1/2(n-2)+n_2/2(2n-1))}$, $N_j \in \mathbb{R}^{(n_1+n)\times 1}$, $L_j \in \mathbb{R}^{3n-4\times((n_1/2)(n-2)+(n_2/2)(2n-1))}$, $K_j \in \mathbb{R}^{3n-4\times 1}$. If (S_2) is used (see (37)), then the matrix M_j is modified accordingly.

One degree of freedom objects: If $n_1 = 2$ the second line of (29) becomes trivial, so one has to use the first line of (29) and modify the constrained equations in consequence (see (37)). Then the first line of (29) becomes

$$q_1^n - q_1^1 = -\sum_{k=1}^{n-1} q_1(k) + \frac{A_2^{i(k+1)}}{A_1^{i(k+1)}} q_2(k+1) + \frac{B^{i(k+1)}}{A_1^{i(k+1)}} \tag{42}$$

and (i′) is modified to the equality

$$\mathscr{A}_{\mathscr{I}_j}(A)x_1 + \mathscr{B}_{\mathscr{I}_j}(A)x_2 + \mathscr{R}_{\mathscr{I}_j}(q_1^1, B) + q_1^n = 0 \quad \text{(i}'\text{)}, \tag{43}$$

where all matrices are constant (in (37) they may generally be function of x_4). One sees that (i′) (ii) (iii) in (36), (37) can be rewritten as

$$M_j x + \bar{N}_j + \begin{pmatrix} q_1^n \\ \mathscr{Q}_{\mathscr{I}_j} q_1^n \\ v_n^+ \end{pmatrix} = 0,$$

where the matrices

$$M_j = \begin{pmatrix} \mathscr{A}_{\mathscr{I}_j} & \mathscr{B}_{\mathscr{I}_j} & 0 \\ \mathscr{C}_{\mathscr{I}_j} & \mathscr{D}_{\mathscr{I}_j} & 0 \\ 0 & 0 & \mathscr{F}_{\mathscr{I}_j} \end{pmatrix} \quad \text{and} \quad \bar{N}_j = \begin{pmatrix} \mathscr{R}_{\mathscr{I}_j} \\ \mathscr{E}_{\mathscr{I}_j} \\ \mathscr{G}_{\mathscr{I}_j} \end{pmatrix}$$

are constant for given initial data. This is the case for the backlash model in Fig. 1. □

3.3. Analytical study of accessibility

The number of impacts $n-1$ is an important parameter in the existence of a solution to Problem 1. This combined with the fact that when $m \geqslant 2$ the index $i(k)$ may vary from one impact to the next, renders the controllability study a hard task in general. However, attacking the problem of multiple impacts as a succession of simple impacts, inherently contains such cumbersome enumeration procedures [13,8]. Note that for Problem 1 to make sense there must exist at least one feasible x^* for (36) and a nonempty open neighborhood of x_4^* in which the entries are nonsingular. Since the denominators are linear in x_4 (consequently the singular set is closed), the matrices entries are even smooth in this neighborhood. Let us state the following:

Lemma 2. *We have*:

(a) *In* (39) $\mathrm{rank}(\mathscr{P}_{\mathscr{I}_j}) = 1$.
(b) *If* (S_1) *is used, then one has* $\mathrm{rank}(\mathscr{M}_{\mathscr{I}_j}) = n_1 - 1$.
(c) *If* (S_2) *is used, one has* $\mathrm{rank}(\mathscr{M}_{\mathscr{I}_j}) = n_1$.

Proof. (a) and (c) follow by direct inspection of the matrices structures, in particular from (37) it follows that

$$\mathscr{M}_{\mathscr{I}_j} = \begin{pmatrix} I_{n_1/2} & 0 \\ \mathscr{Q}_{\mathscr{I}_j} & 0 \\ 0 & I_{n_1/2} \end{pmatrix}.$$

(b) In (30) one has $\mathrm{rank}(\mathscr{R}_{\mathscr{I}_j}) = \frac{n_1}{2} - 1$. Indeed if v is an eigenvector of $\mathscr{R}_{\mathscr{I}_j}$ and λ the corresponding eigenvalue, one has $\mathscr{R}_{\mathscr{I}_j} v = \lambda v$. It follows that $\frac{\dot{q}_1(t_{n-1}^+)A_1^{i(n)}}{A_1^{i(n)}\dot{q}_1(t_{n-1}^+)} v = (1-\lambda)v$. The matrix $\frac{\dot{q}_1(t_{n-1}^+)A_1^{i(n)}}{A_1^{i(n)}\dot{q}_1(t_{n-1}^+)}$ has rank 1, and it is easy to see that all its eigenvalues are 0 except one eigenvalue equal to 1, whose eigenvector is $v = \dot{q}_1(t_{n-1}^+)$. Therefore, $\mathscr{R}_{\mathscr{I}_j}$ has all eigenvalues equal to 1 except one that is 0. The result then follows from the structure of the matrix $\mathscr{Q}_{\mathscr{I}_j}$ in (35) and that of $\mathscr{M}_{\mathscr{I}_j}$. $\square$

Lemma 3. *Accessibility in the sense of Definition* 2 *implies* $\frac{4n_1+n_2}{n_1+2n_2} \leqslant n$.

Proof. The proof can be done by direct calculation and inspection of the structure of the matrices M_j and N_j in (41). One has $\mathrm{rank}(\mathscr{M}_{\mathscr{I}_j}) \leqslant n_1$. In the general case of n impacts, one has $M_j \in \mathbb{R}^{(n_1+n)\times((n_1/2)(n-2)+(n_2/2)(2n-1))}$. Accessibility implies that for a given (q_1^1, v_1^+), the equality in (39) be satisfied for x_n in an open set. Consequently, it must hold that $\mathrm{Im}(M_j) \supseteq \mathrm{Im}(\mathscr{M}_{\mathscr{I}_j})$. The representations in (S_1) and (S_2) being equivalent one to each other, let us use (S_2). It follows that $(\mathrm{rank}(M_j) \geqslant \mathrm{rank}(\mathscr{M}_{\mathscr{I}_j})) \Leftrightarrow \frac{n_1}{2}(n-2) + \frac{n_2}{2}(2n-1) \geqslant n_1 \Longleftrightarrow n \geqslant \frac{4n_1+n_2}{n_1+2n_2}$. $\square$

Clearly, the lower bounds in Lemma 3 are conservative and should be refined. Let us propose the following, without proof:

Conjecture 1. *One has* $n_1 \leqslant 2^n$ *if accessibility holds.*

3.3.1. The case $n = 2$

Controllability as in Definition 4 holds if there exists n such that $\bar{\mathscr{R}}^n[(q_1^1, v_1^+)] = \mathbb{R}^{n_1/2} \times \mathbb{R}^{n_1/2}$ for any (q_1^1, v_1^+). One may examine the structure of $\mathscr{R}^2[(q_1^1, v_1^+)]$ in order to determine whether or not a covering of $\mathbb{R}^{n_1/2} \times \mathbb{R}^{n_1/2}$ is possible in a finite or infinite number of impacts (more exactly one should speak of the object configuration space $\times$ its tangent space). The reason may be that the study of the spaces $\mathscr{R}^2[(q_k^1, v_k^+)]$ may be more tractable, hence the characterization of $\mathscr{R}^2[(q_k^1, v_k^+)]$, $(q_k^1, v_k^+) \in \mathscr{R}^2[(q_{k-1}^1, v_{k-1}^+)]$, may be useful. Let us therefore better understand the constrained nonlinear equation (36) when $n = 2$. Note first that x_1 no longer appears in the problem, so that $x^{\mathrm{T}} = (x_2^{\mathrm{T}}, x_4^{\mathrm{T}})$, $x \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_2/2}$. This means that the matrix $\mathscr{A}_{\mathscr{I}_j}$ is no longer defined in both (30) and (37). The structure of all the matrices defined in (39), (40) and (41), has to be modified accordingly.

Lemma 4. *If* $n = 2$ *in Problem* 1, *then the matrices* M_j, N_j, L_j, K_j *in* (41) *are constant* (*i.e. they do not depend on* x_4), *and* (iv) *in* (36) *simplifies to an inequality* $\mathscr{J}_{\mathscr{I}_j}(q_1^1, q_1^2, v_1^+, A) \geqslant 0$. *Moreover*, $\mathrm{rank}(M_j) = 3$ *and accessibility in one impact implies* $n_1 = 2$.

Proof. Here, we use the formalism (S_2), constructed from (37). The proof is done by inspection of the structure of the matrices in (36), replacing (i) by (i') in (37). In particular, one has

$$M_j = \begin{pmatrix} 0 & \dfrac{v_1^+ A_2^{i(2)}}{A_1^{i(2)} v_1^+} & 0 \\ A_2^{i(1)} & 0 & 0 \\ 0 & A_2^{i(2)} & 0 \\ 0 & 0 & \dfrac{(1+e) N_1^{i(2)} A_2^{i(2)}}{A^{i(2)} M^{-1} (A^{i(2)})^{\mathrm{T}}} \end{pmatrix} \in \mathbb{R}^{(n_1+2)\times(3/2)n_2}.$$

The rank follows as there are three columns which are independent vectors in $\mathbb{R}^{n_1+2}$. Using the formalism in (41) constructed from (37) (in other words, using the matrix $\mathscr{M}_{\mathscr{I}_j}$ of (39) for (S_2)), one sees that $\mathrm{rank}(\mathscr{M}_{\mathscr{I}_j}) = n_1$. For the system to be accessible, Problem 1 has to have solutions for (q_1^2, v_2^+) evolving in a set containing an open set of $\mathbb{R}^{n_1}$. From (41) this implies that $\mathrm{rank}(M_j) \geqslant \mathrm{rank}(\mathscr{M}_{\mathscr{I}_j})$. Consequently, $n_1 \leqslant 3$, i.e. $n_1 = 2$. $\square$

It is useful to think of Problem 1 (i.e. of (36)) in terms of a convex quadratic program, which in turn can be posed as a LCP [23, Section 1.3.4]. Some definitions concerning LCPs are given in Appendix A.

Corollary 1. *If $n = 2$, Problem* 1 *has a solution only if the mixed linear complementarity problem* (mLCP) *in* (46) *is solvable.*

Proof. First, let us note that since $n = 2$, the inequality constraint (vi) reduces to $A_1^{i(2)} v_1^+ > 0$ or < 0. We therefore disregard it. If x^* is a solution of Problem 1, then from (41) it is also a solution of the quadratic programme (QP)

$$\min \tfrac{1}{2}(M_j x + N_j)^{\mathrm{T}}(M_j x + N_j),\ \mathscr{K}_{\mathscr{I}_j} + \mathscr{L}_{\mathscr{I}_j} x_4 \geqslant 0. \tag{44}$$

Hence the Karush–Kuhn–Tucker (KKT) necessary and sufficient conditions are satisfied, i.e. there exists a slack variable $\mu \in \mathbb{R}$ such that

$$\begin{cases} M_j^{\mathrm{T}} M_j x^* + M_j^{\mathrm{T}} N_j - L_j^{\mathrm{T}} \mu = 0, \\ 0 \leqslant \mu \perp \mathscr{K}_{\mathscr{I}_j} + \mathscr{L}_{\mathscr{I}_j} x_4 \geqslant 0, \end{cases} \tag{45}$$

where $L_j = (0\ \ \mathscr{L}_{\mathscr{I}_j}) \in \mathbb{R}^{1\times(3/2)n_2}$. If x^* is a KKT point of this program [23, Section 9.3.1] and in addition $M_j x^* + N_j = 0$, then (q_1^2, v_2^+) is reachable from (q_1^1, v_1^+). If one can show that there is a set in $\mathbb{R}^{n_1/2} \times \mathbb{R}^{n_1/2} \ni (q_1^2, v_2^+)$ containing an open set, and such that both conditions are satisfied, then the system is reachable in one impact. We know from Lemma 4 that $\mathrm{rank}(M_j) = 3$. If $M_j^{\mathrm{T}} M_j \in \mathbb{R}^{(3/2)n_2 \times (3/2)n_2}$ has full-rank 3 (this is the case only if $n_2 = 2$), one can easily transform (45) into a LCP(μ) with a positive definite matrix. If $n_2 \geqslant 4$, let us denote $\bar{r} = \frac{3}{2} n_2 - 3$. Let us define $E_3 = (0\ I_3)$, $E_{\bar{r}} = (I_{\bar{r}}\ 0)$ and W, $W^{\mathrm{T}} W = I_{(3/2)n_2}$, such that $W M_j^{\mathrm{T}} M_j W^{\mathrm{T}} = \mathrm{diag}(0_{\bar{r}\times\bar{r}}, D_j)$,where $D_j > 0$ is diagonal and 3×3. Then (45) can be rewritten as

$$\begin{cases} D_j E_3 W x^* + E_3 W M_j^{\mathrm{T}} N_j - E_3 W L_j^{\mathrm{T}} \mu = 0, \\ E_{\bar{r}} W M_j^{\mathrm{T}} N_j - E_{\bar{r}} W L_j^{\mathrm{T}} \mu = 0, \\ 0 \leqslant \mu \perp \mathscr{K}_{\mathscr{I}_j} + \mathscr{L}_{\mathscr{I}_j} x_4^* \geqslant 0 \end{cases} \tag{46}$$

which is a mLCP. $\square$

One may solve the mLCP numerically, and then look for solutions which satisfy $M_j x^* + N_j = 0$. In the case $n = n_1 = n_2 = 2$, one may use LCPs to study accessibility and controllability, see Section 4. The use of complementarity problems is important because it paves the way to theoretical and numerical studies.

Corollary 2. (a) $(q_1^2, v_2^+) \in \mathscr{R}^2[(q_1^1, v_1^+)] \Longrightarrow$ (b) $\mathscr{I}_{\mathscr{I}_j}(q_1^1, q_1^2, v_1^+) \geqslant 0$ *and* $x^*(0)$ *is a KKT point of the QP in* (44) $\Longleftrightarrow$ (c) $\mathscr{I}_{\mathscr{I}_j}(q_1^1, q_1^2, v_1^+) \geqslant 0$ *and the mLCP in* (46) *is solvable with* $\mu = 0$. *If* $\mathrm{Ker}(M_j^{\mathrm{T}}) = \{0\}$ *then* (a) $\Longleftrightarrow$ (b).

Proof. Simply note that if (q_1^2, v_2^+) is reachable from (q_1^1, v_1^+) then $M_j x^* + N_j = 0$ so that $\mu = 0$, and $\mathscr{H}_{\mathscr{I}_j}$ in (33) vanishes. $\square$

3.3.2. The general case

A vector x is said admissible if the entries of the matrices in (36) are bounded at x. The set of admissible x is open.

Lemma 5. *Consider the constrained equation* (S_2) *in* (39). *The system is accessible in* $(n-1)$ *impacts in the sense of Definition* 2 *if and only if there exists a set V of admissible x and* $j \in \{1, \ldots, m^{(n-1)}\}$ *such that*

(a) $-[\mathscr{P}_{\mathscr{I}_j}(x_4), 0]\ (\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{M}_{\mathscr{I}_j})^{-1}\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{N}_{\mathscr{I}_j}(x) + \mathscr{T}_{\mathscr{I}_j}(x) \geqslant 0$ *for all* $x \in V$,

(b) $\mathscr{K}_{\mathscr{I}_j} + \mathscr{L}_{\mathscr{I}_j}x_4 \geqslant 0$ *and* $\mathscr{U}_{\mathscr{I}_j}v_1^+ + \mathscr{V}_{\mathscr{I}_j}x_4 + \mathscr{W}_{\mathscr{I}_j} > 0$ *or* <0 *for all* $x \in V$,

(c) $\mathscr{N}_{\mathscr{I}_j}(x) \in \mathrm{Ker}\ [I_{n_1+n} - \mathscr{M}_{\mathscr{I}_j}(\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{M}_{\mathscr{I}_j})^{-1}\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}]$ *for all* $x \in V$,

(d) $\bigcup_{j\in\{1,\ldots,m^{(n-1)}\}} \mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\ \mathrm{span}_{x\in V}\ [\mathscr{N}_{\mathscr{I}_j}(x)]$ *contains an open set of* $\mathbb{R}^{n_1}$.

Proof. We suppress the arguments in the matrices for the sake of clarity of the writing. Let us first show that if the system is accessible in $(n-1)$ impacts, then (a)–(d) hold. Accessibility implies that there exists at least one x_n and one j such that (S_2) written as in (39) is satisfied for some x. One has $\mathscr{M}_{\mathscr{I}_j}x_n + \mathscr{N}_{\mathscr{I}_j} = 0 \Rightarrow x_n = (\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{M}_{\mathscr{I}_j})^{-1}\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{N}_{\mathscr{I}_j}$. Thus, (a) follows from $\mathscr{P}_{\mathscr{I}_j}x_n + \mathscr{T}_{\mathscr{I}_j} \geqslant 0$. (b) is just (v), (vi) in (39). Next, we have

$$\begin{aligned}
\mathscr{M}_{\mathscr{I}_j}x_n + \mathscr{N}_{\mathscr{I}_j} = 0\ &\Rightarrow \mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}[\mathscr{M}_{\mathscr{I}_j}x_n + \mathscr{N}_{\mathscr{I}_j}] = 0\\
&\Rightarrow x_n + (\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{M}_{\mathscr{I}_j})^{-1}\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{N}_{\mathscr{I}_j} = 0\\
&\Rightarrow \mathscr{M}_{\mathscr{I}_j}x_n + \mathscr{M}_{\mathscr{I}_j}(\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{M}_{\mathscr{I}_j})^{-1}\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{N}_{\mathscr{I}_j} = 0\\
&\Rightarrow -\mathscr{N}_{\mathscr{I}_j} + \mathscr{M}_{\mathscr{I}_j}(\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{M}_{\mathscr{I}_j})^{-1}\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{N}_{\mathscr{I}_j} = 0\\
&\Rightarrow \mathscr{N}_{\mathscr{I}_j} \in\ \ \mathrm{Ker}\ [I_{n_1+n} - \mathscr{M}_{\mathscr{I}_j}(\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{M}_{\mathscr{I}_j})^{-1}\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}]
\end{aligned} \tag{47}$$

so V is not void and (c) is proved. Finally, $x_n = -(\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{M}_{\mathscr{I}_j})^{-1}\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{N}_{\mathscr{I}_j}$ and since accessibility implies that (S_2) is satisfied for x_n in an open set of $\mathbb{R}^{n_1}$, condition (d) follows remembering that $\mathscr{M}_{\mathscr{I}_j}$ has constant rank n_1.

Let us prove now that if (a)–(d) hold, then the system is accessible in $(n-1)$ impacts. Equivalently, there exists an open set $\mathscr{O} \subset \mathbb{R}^{n_1}$ such that (S_2) is satisfied for all $x_n \in \mathscr{O}$ (i.e. there is a set V such that for each x_n in $\mathscr{O}$, (S_2) is satisfied with some $x \in V$ and some $j \in \{1, \ldots, m^{(n-1)}\}$). Let us consider (c). It implies that $\mathscr{N}_{\mathscr{I}_j} = \mathscr{M}_{\mathscr{I}_j}(\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{M}_{\mathscr{I}_j})^{-1}\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{N}_{\mathscr{I}_j}$, and the vector $x_n = -(\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{M}_{\mathscr{I}_j})^{-1}\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{N}_{\mathscr{I}_j}$ is such that $\mathscr{M}_{\mathscr{I}_j}x_n + \mathscr{N}_{\mathscr{I}_j} = 0$. Considering now (d), one sees that as x varies in V and j varies through $j \in \{1, \ldots, m^{(n-1)}\}$, then $\mathscr{M}^{\mathrm{T}}_{\mathscr{I}_j}\mathscr{N}_{\mathscr{I}_j}(x)$ spans an open set of $\mathbb{R}^{n_1}$. Thus, x_n can be taken in an open set. Condition (a) then implies that (iv) of (S_2) holds in this open set, and (b) allows one to get (v), (vi).

If one uses (S_1), then similar conditions can be derived taking into account that $\mathrm{rank}(\mathscr{M}_{\mathscr{I}_j}(x_4)) = n_1 - 1$. $\square$

Lemma 5 can be used to test the nonaccessibility by checking that one of the conditions (a)–(d) fails.

It is clear that a next step is to study how the physical parameters of the juggler (namely e, A, B, M, n_1, n_2) influence the existence of solutions to the various constrained equations that characterize the reachable subspaces.

4. Example: dynamic backlash model

As an illustration, let us consider the impacting pair in Fig. 1. Since $n_1 = 2$, the comment on one degree-of-freedom objects in Section 3.2 applies. The following is true.

Lemma 6. *The dynamics of the impacting pair's object is accessible in* 1 *impact in the sense of Definition* 2.

The proof is given in Appendix B. It is an application of Corollary 2, however we develop all the calculations and provide an accurate characterization of the reachable subspaces.

Lemma 6 is a first step to show the following:

Lemma 7. *The dynamics of the impacting pair's object is controllable in* 3 *impacts in the sense of Definition* 4.

The proof is given in Appendix C. It relies on the characterization of the spaces $\mathscr{R}^2[(q_1^k, v_k^+)]$ which can be analytically computed. Both Lemmas 6 and 7 hold independently of $e \in [0, 1]$. The shape of the reachable subspaces for the impacting pair that is found from the analysis in Appendices B and C is intuitively sound (see Fig. 4).

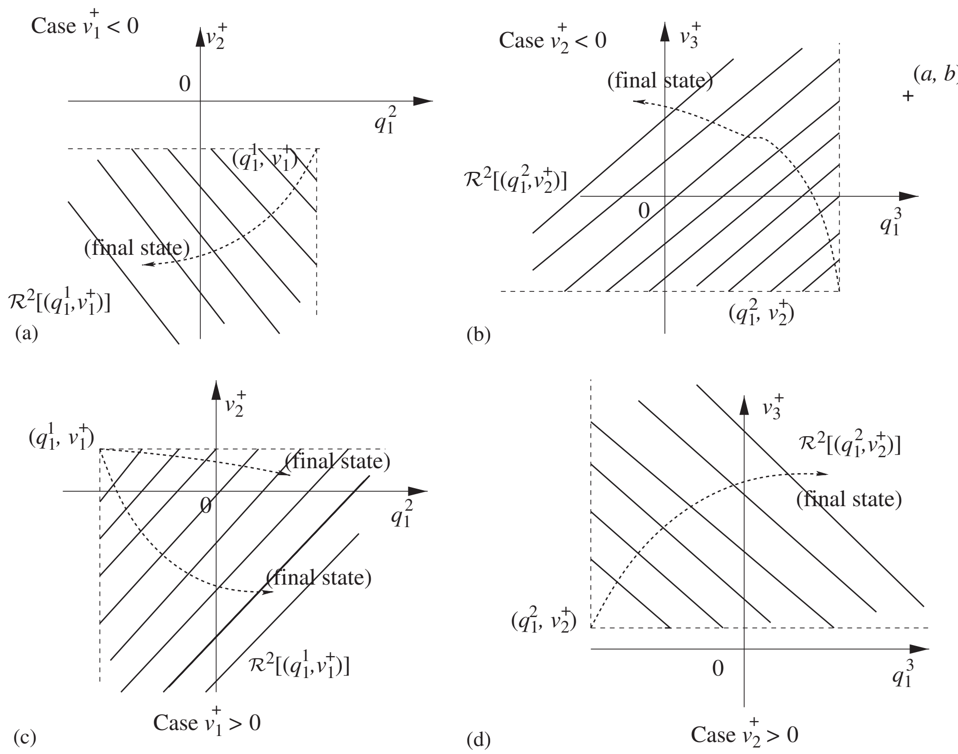

Fig. 4. Reachable subspaces for the backlash model.

5. Conclusions

This work focuses on the controllability of a class of nonsmooth complementarity mechanical systems, named jugglers because of their particular dynamics. Potential and important applications can be found in nonprehensile manipulation, kinematic chains with dynamic backlash, controlled structures, manipulators with dynamic passive obstacle, hopping and running machines, vibro-impact processes, etc. The simplest jugglers are examined, which anyway remain highly nonlinear systems. It is shown that the attainable subspaces of the object dynamics are characterized by constrained equations. These constrained equations are shown to possess a specific structure so that preliminary analytical results can be derived. The usefulness of the developed analysis is twofold: firstly the presented tools, which heavily rely on complementarity problems, pave the way towards numerical computation of the reachable subspaces. Secondly, jugglers may be designed so that the complementarity problems characterizing their reachable subspaces, are well-posed. Extension towards systems with friction, nonlinear constraints, objects subject to gravity, is the topic of future works.

Acknowledgements

This work was partially supported by the European project SICONOS IST-2001-37172. We would like to thank an anonymous reviewer for his/her very careful reading of our paper.

Appendix A. Complementarity problems

Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$, the problem of finding $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ satisfying

$$y = Ax + B \geqslant 0, \quad x \geqslant 0, \quad x^{\mathrm{T}} y = 0 \tag{48}$$

is called a linear complementarity problem (LCP) [3]. It can be equivalently written as

$$0 \leqslant x \perp y = Ax + B \geqslant 0. \tag{49}$$

Roughly, the LCP has a unique solution x^* whatever B if and only if A satisfies some positivity conditions, see [23]. Positive definiteness of A is sufficient.

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times n}$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, the problem of finding $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ satisfying

$$a + Ax + Cy = 0, \quad 0 \leqslant b + Dx + By \perp y \geqslant 0 \tag{50}$$

is called a mLCP [3].

Appendix B. Accessibility in one impact

Let us consider the case $n_1 = n_2 = n = 2$. Then if Problem 1 has a solution x^* and using notations in Lemma 4, this must be the value that minimizes the quadratic function

$$\Psi(x) = \frac{1}{2}\left[M_j x + \bar{N}_j + \begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)} q_1^2 \\ v_2^+ \end{pmatrix}\right]^{\mathrm{T}} \left[M_j x + \bar{N}_j + \begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)} q_1^2 \\ v_2^+ \end{pmatrix}\right]$$

subject to $L_j(x_4)x + K_j(x_4) \geqslant 0$, and $x = (q_2(1), q_2(2), \dot{q}_2(2))^{\mathrm{T}} \in \mathbb{R}^{3\times 1}$. The matrices $M_j \in \mathbb{R}^{4\times 3}$ and $N_j \in \mathbb{R}^{4\times 1}$ are constant, see the last item in Section 3.2. The inequality in (S_1) reduces to (v), i.e. from (20): $-A_1^{i(2)} v_1^+ - A_2^{i(2)} x_4 \geqslant 0$, because (iv) disappears from the analysis in a first stage. The KKT necessary and sufficient conditions stipulate the existence of a Lagrange multiplier $\mu \in \mathbb{R}$ such that [23, Section 9.3.1]

$$\begin{cases} M_j^{\mathrm{T}} M_j x^* + M_j^{\mathrm{T}} \bar{N}_j + M_j^{\mathrm{T}} \begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)} q_1^2 \\ v_2^+ \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ A_2^{i(2)} \end{pmatrix} \mu = 0, \\ \mu \geqslant 0, \quad \mu^{\mathrm{T}}(A_2^{i(2)} x_4^* + A_1^{i(2)} v_1^+) = 0, -A_2^{i(2)} x_4^* - A_1^{i(2)} v_1^+ \geqslant 0 \end{cases} \tag{51}$$

for a $j \in \{1, 2\}$ (in other words, one can choose to strike with either constraint at time t_2 and to initialize the system with any constraint at time t_1). Since $\mathrm{rank}(M_j) = 3$ (see Lemma 4), the second line in (51) is a LCP(μ) with LCP-matrix (a scalar in this case)

$$M_{LCP} = (0 \quad 0 \quad A_2^{i(2)})\,(M_j^{\mathrm{T}} M_j)^{-1} \begin{pmatrix} 0 \\ 0 \\ A_2^{i(2)} \end{pmatrix} > 0.$$

Therefore, LCP(μ) always possesses a unique solution [23, Theorem 3.13]. From the first line of (51) one has

$$x^*(\mu) = (M_j^{\mathrm{T}} M_j)^{-1} \left[-M_j^{\mathrm{T}} \bar{N}_j - M_j^{\mathrm{T}} \begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)} q_1^2 \\ v_2^+ \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ A_2^{i(2)} \end{pmatrix} \mu \right]. \tag{52}$$

Let us come back on the backlash model, where m_1 and m_2 are the masses of the two bodies in Fig. 1. If we choose to strike at t_1 the constraint 2 and at t_2 the constraint 1 (i.e. $i(1) = 1$ and $i(2) = 2$), we have $A_1^{i(2)} = A_1^2 = -1$ and $A_2^{i(2)} = A_2^2 = 1$. Consider x_4^* calculated from (52) with $\mu = 0$, and let us denote it as $x_4^*(0)$. So if $x_4^*(0) - v_1^+ < 0$ the solution of LCP(μ) is $\mu = 0$. Injecting (52) into the left-hand side of this inequality, one sees that the obtained scalar function is a linear function of (q_1^2, v_2^+) and since rank$(M_j)=3$, there is a half space in the (q_1^2, v_2^+)-plane such that $x_4^*(0)-v_1^+<0$. More precisely, in this case one has

$$M_j^{\mathrm{T}}\begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)}q_1^2 \\ v_2^+ \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{-(1+e)m_2}{m_1+m_2} \end{pmatrix}\begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)}q_1^2 \\ v_2^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{-(1+e)m_2}{m_1+m_2}v_2^+ \end{pmatrix}.$$

Therefore, $x_4^*(0)$ is a linear function of v_2^+ only. In addition to be a KKT point of the QP for the suitable values of v_2^+ in a half-plane, x^* in (52) and with $\mu=0$ satisfies the equality

$$M_j x^* + N_j + \begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)}q_1^2 \\ v_2^+ \end{pmatrix} = 0.$$

Let us examine now inequality (iv). From (32) one finds $q_1^2-q_1^1=v_1^+\Delta_1$. The constraint $\Delta_1\geqslant 0$ therefore yields $q_1^2\leqslant q_1^1$ if $v_1^+\leqslant 0$, and $q_1^2\geqslant q_1^1$ if $v_1^+\geqslant 0$. We conclude about accessibility of the impacting pair in one impact since $\bar{\mathscr{R}}^2[(q_1^1, v_1^+)]=\mathscr{R}^2[(q_1^1, v_1^+)]$ is a quadrant in the (q_1^2, v_2^+)-plane, as intuitively expected (see Fig. 4).

Appendix C. Controllability in three impacts

The proof relies on the following arguments:

- The goal is to show that for any initial data (q_1^1, v_1^+) one can reach any (q_1^4, v_4^+) after three impacts. In Appendix B, we have shown that $\mathscr{R}^2[(q_1^1, v_1^+)]=[q_1^1, +\infty)\times[-\infty, v_1^+)$ if $v_1^+>0$ and $\mathscr{R}^2[(q_1^1, v_1^+)]=(-\infty, q_1^1]\times[-\infty, v_1^+)$ if $v_1^+<0$.
- Let us choose in a second step a sequence of impacts, that comes after the second impact, that consists of a third impact at time t_3 and with the constraint number 1. In other words, we restart the analysis and we choose $i(1)=2$ and $i(2)=1$. The new initial condition for this second sequence is (q_1^2, v_2^+), and the final condition is (q_1^3, v_3^+). This time calculations show that $\mathscr{R}^2[(q_1^2, v_2^+)]=(-\infty, q_1^2]\times[v_2^+, +\infty)$ if $v_2^+<0$, and $\mathscr{R}^2[(q_1^2, v_2^+)]=[q_1^2, +\infty)\times[v_2^+, +\infty)$ if $v_2^+>0$.
- The reachable sets which correspond to such a succession of impacts are depicted in Fig. 4. One can easily imagine what the sets look like if a third impact at t_4 is imposed, with the same sequence of attained surfaces as in the first case. The idea is therefore to prove controllability by looking for sets $\mathscr{R}^2[(q_1^k, v_k^+)]$ and $(q_1^k, v_k^+)\in\mathscr{R}^2[(q_1^{k-1}, v_{k-1}^+)]$.
- Inspection of the reachable sets in the four cases depicted in Fig. 4 shows that whatever the final state, one can always choose a combination of impacted surfaces in order to attain such a point of the object's state space. Consider, for instance, the point (a, b) in Fig. 4(b), with $|a|>|q_1^1|$ and $v_1^+<0$. It is not possible to reach this point in two impacts (Fig. 4(a) and (b)). However, choosing $v_2^+>b$ one can reapply an initial sequence with positive velocity (Fig. 4(c)) and reach the desired point.
- We conclude that it is possible to reach the whole plane after at most three impacts, i.e. $\bar{\mathscr{R}}^4[(q_1^1, v_1^+)]=\mathscr{R}^2[(q_1^1, v_1^+)]\cup\mathscr{R}^3[(q_1^1, v_1^+)]\cup\mathscr{R}^4[(q_1^1, v_1^+)]=\mathbb{R}^2$, which allows us to conclude about controllability in three impacts of the impacting pair's object. Possibly the system is controllable in less impacts than 3, because we have not checked all possible sequences between the set of indices $(1, 2)$ and the initial velocity sign (i.e. we have not constructed all the possible constrained equations in (36)).

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