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Contact Stability Analysis of a One Degree-of-Freedom Robot

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Abstract. The aim of this note is to examine the conditions of stability of a simple robotic task: we consider a one degree-of-freedom (dof) robot that collides with a spring-like environment with stiffness k, the goal being to stabilize the system in contact with the environment. We study conditions on the feedback gains that guarantee quadratic Lyapunov stability of the task with a well-conditioned solution to the Lyapunov equation. It is shown that when the environment's stiffness k grows unbounded, those conditions yield unbounded values of the gains. Motivated by the stability analysis of the impact Poincaré map in the perfectly rigid case ($k = +\infty$), we propose an analysis that is independent of k. It enables us to conclude on global asymptotic convergence of the system's state towards the equilibrium point. This work can also be seen as the study of stability of a contact (force control) phase, taking into account the *unilateral* feature of the constraint.

Keywords: free and constrained motion, compliant and rigid environment, quadratic Lyapunov stability, contact stability, impact Poincaré map

1. Introduction

The last fifteen years have witnessed important progresses in the theory of manipulator's control. It has been assumed that the robot evolves either in a free space (motion control), or that it remains in contact with a certain environment (hybrid force/position control). Both cases have been considered separately in the literature, and several solutions have been proposed in each case. However a real robotic task often implies phases of transition between free motion and constrained motion, and the so-called *transition phase* appears to be in most cases crucial for the system's stability. This is the case for hopping robots, walking machines, and manipulation with a robotic hand. Some interesting results for the transition phase control have been presented, see e.g. [21] [25] [18] [19]. Also many studies have been devoted to investigate the so-called *contact stability* problem, due to the unilaterality of the robot's tip constraint [7] [24] [5] [1] [8]. It seems that Whitney [24] was the first to point out and analyze such problems. However the first papers dealing with the transition phase containing a complete stability analysis for a n-degree of freedom robot controlled by a switching algorithm were recently proposed in [12] [3].

The goal of this study is not to extend these works to more complex environments models or control algorithms, but rather to point out some problems related with stabilization of motion-controlled manipulators that come in contact with a compliant environment, in particular the sufficient conditions guaranteeing asymptotic convergence of the solutions towards the steady-state solution. Indeed we restrict ourselves to a simple continuous PD motion controller (in contrast with the sophisticated switching controllers studied in [3] [10] [12] [11]) and to the case of a purely elastic environment. Basically our motivation is the following: a question a designer may ask himself when facing a real problem is: should the environment be considered as flexible or as rigid? In general one considers that rigid body collisions occur when the bodies show "sufficiently small" deformation so that they are geometrically rigid at a global observation scale [14]. The answer is crucial for the choice of the control algorithm[1] and depending on it, the analysis of the whole robotic task may be quite different. Indeed rigid body dynamics involve models which completely differ from those used when compliant bodies are considered (i.e. respectively *measure differential equations* [13] [14] instead of *ordinary differential equations*). In the rigid body impacting case, it is customary and convenient to study the so-called *impact Poincaré map* which is a discrete-time reduced-order system [23]. In the compliant body case (in general spring-dashpot contact models), one usually directly uses a Lyapunov-like stability analysis, the system being in this case a very simple example of a *hybrid dynamical system* [2]. Note that the boundary between "flexible" and "rigid" is quite clear from a mathematical point of view, but not from a practical one: Besides clearly rigid environments made of hard materials (concrete, iron ...) and clearly flexible ones, some others might be considered to belong to one class or the other one depending on the task (masses of the bodies that collide, accuracy of the measurements, limits of the actuators ...). We believe the results in this paper may help in partially answering this question.

The note is organized as follows: in section 2, the system and the notations are introduced and we show with a particular Lyapunov function that the closed-loop system's fixed point is Lyapunov globally asymptotically stable (GAS) for a suitable choice of the feedback gains, and for $k < +\infty$. In section 3 we analyze the problem related with quadratic stability of the task when the environment's stiffness grows unbounded. We show that in order to guarantee that the solution P to the system's Lyapunov equation satisfies $\lambda_{\min} P \geq \delta > 0$ for some δ, where $\lambda_{\min}$ denotes the smallest eigenvalue, the feedback gains grow unbounded as $k \to +\infty$. Section 4 is devoted to analytically prove global asymptotic convergence of the position and velocity tracking errors towards zero, independently of the contact stiffness value. The analysis is shown to reduce to the stability analysis of the impact Poincaré map associated to the closed-loop system when $k = +\infty$. Finally conclusions are drawn in section 5, and some technical results are developed and recalled in the appendices.

2. A Simple Example

The system consists of a simple mass moving horizontally without friction whose position is given by $x(t)$, and a compliant environment at $x = 0$ whose model is a spring with stiffness $k > 0$ (see figure 1). The control law is given by $u = -\lambda_2 \dot{x} - \lambda_1 (x - x_d)$, $x_d \geq 0$, $\lambda_1 > 0$, $\lambda_2 > 0$. We assume that contact is established at $t = 0$, with $x = 0$. Then the equations that govern our system are:

$$\begin{cases} m\ddot{x} + \lambda_2 \dot{x} + \lambda_1 x = \lambda_1 x_d & \text{if } x < 0 \\ m\ddot{x} + \lambda_2 \dot{x} + (\lambda_1 + k)x = \lambda_1 x_d & \text{if } x \geq 0 \end{cases} \tag{1}$$

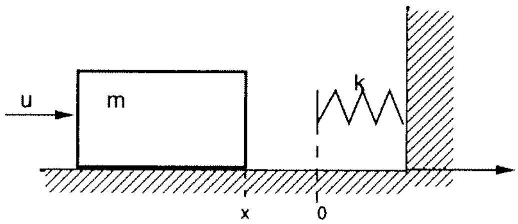

Figure 1. Controlled mass colliding an elastic wall.

Notice that convergence of the state $(x, \dot{x})$ towards the fixed point of the second equation in (1) may be investigated by considering the associated equivalent mechanical system to the closed-loop system in (1). Notice however that the equivalent total mechanical energy for such a nonlinear discontinuous system is not simply the concatenation of both the (closed-loop) energy functions for the free and contact phases. Indeed it corresponds to the so-called *available storage function* in dissipative systems theory. Calculations can be found in [4]. In the sequel we focus on a particular stability property of that equilibrium point. The motivation for studying this type of stability is evident if one thinks of more complicated tasks as considered for instance in [12]. Also the equivalence with a mechanical system may no longer be possible in certain cases, e.g. when the feedback loop contains time-delays or for tracking control.

When the contact stiffness is finite, one can treat such a stabilization problem from different point of views, for instance: i) Study conditions that guarantee that after the first contact has occured, there is no rebound [10], ii) Relax the bounceless conditions by studying conditions that insure Lyapunov quadratic stability of the system, i.e. find a Lyapunov function $V(x)$ such that along trajectories of the system $\dot{V}(x) = -x^T Qx$ with $Q > 0$ (which does not *a priori* guarantee that the robot's tip will never take off the environment's surface), see e.g. [12]. Since these tools will generally provide sufficient conditions only, it is worth investigating whether these conditions are of any practical interest or not. In particular, if they yield lowerbounds on the feedback gains that are proportional to the environment's stiffness, it is clear that as soon as this stiffness becomes too large, the conditions become useless. It is then natural to seek a convergence proof that is independent of the stiffness as well as of the feedback gains values. For the sake of briefness of the paper, we shall study only approach ii). In fact it can be easily (and logically) concluded that bounceless conditions are impossible to obtain with finite force control, for nonzero contact velosity, as the stiffness k grows unbounded. It is note worthy that this work can also be seen as a study on conditions of stability of a force control scheme, taking into account the fact that the constraints are unilateral, i.e. the robot's tip may take off the surface and possibly start a sequence of rebounds.

Before going on with the stability analysis of system (1) when $k < +\infty$, let us recall that as $k \to +\infty$, the solutions of (1) converge towards the solutions of the following dynamical system:

$$\begin{cases} m\ddot{x} + \lambda_2\dot{x} + \lambda_1 x = \lambda_1 x_d & \text{if } x < 0 \\ \dot{x}(t_k^+) = -\dot{x}(t_k^-) & \text{if } x(t_k) = 0 \\ \ddot{x} = \min(0, -\lambda_2\dot{x} - \lambda_1 x + \lambda_1 x_d) & \text{if } \dot{x}(t_k^+) = 0 \end{cases} \tag{2}$$

The t_k's generically denote the impact times. The proof of convergence can be found in [15], together with a rigorous definition of the used notion of convergence. It is therefore legitimate to seek a stability analysis that encompasses both systems in (1) and (2), i.e. that works for all $k \in [0, +\infty]$. We shall come back on the definition and on the stability analysis of the impact Poincaré map associated to (2) in section 4.

For the moment we shall analyze the stability of the task using a single Lyapunov function. To begin with, we show how the stability analysis of the closed-loop system in (1) can be led with a particular Lyapunov function candidate: Let us consider

$$V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\lambda\tilde{x}^2 + c\tilde{x}\dot{x} \tag{3}$$

with $\lambda = \lambda_1 + k + \frac{\lambda_2 c}{m}$, $c > 0$ is such that $c^2 - \lambda_2 c - m(\lambda_1 + k) < 0$ (since $\Delta = \lambda_2^2 + 4m(\lambda_1 + k) > 0$, and $\sqrt{\Delta} - \lambda_2 > 0$, such a c can always be chosen arbitrarily small), and $\tilde{x} = x - \frac{\lambda_1 x_d}{\lambda_1 + k}$. λ and c guarantee that V is positive definite. Now we get along closed-loop trajectories:

- $x < 0$ (non-contact)

$$\begin{aligned} \dot{V} \le & \left(-\lambda_2 + c + \frac{1}{2}k^2 + \frac{1}{2}\right)\dot{x}^2 + \left(-\frac{c\lambda_1}{m} + 1\right)\tilde{x}^2 + \frac{1}{2}\left(\frac{\lambda_1 k}{\lambda_1 + k}x_d\right)^2 \\ & + \frac{1}{2}\left(\frac{c\lambda_1 k}{(\lambda_1 + k)m}x_d\right)^2 \end{aligned} \tag{4}$$

or in compact form

$$\dot{V} = -a_{nc}\dot{x}^2 - b_{nc}\tilde{x}^2 + R \tag{5}$$

with $R > 0$.

- $x > 0$ (contact)

$$\dot{V} = (-\lambda_2 + c)\dot{x}^2 - \frac{\lambda_1 + k}{m}c\tilde{x}^2 = -a_c\dot{x}^2 - b_c\tilde{x}^2 \tag{6}$$

Claim 1. For any stiffness $0 < k < +\infty$ there exist $P = P^T > 0$, $Q = Q^T > 0$, $\lambda_1^\star < +\infty$, $\lambda_2^\star < +\infty$ such that $\lambda_1 > \lambda_1^\star$, $\lambda_2 > \lambda_2^\star$ implies that for all $t \ge 0$, $V = z^T P z$, $\dot{V} \le -z^T Q z$. Thus the equilibrium point $z = 0$ is globally asymptotically stable (GAS).

The proof is given in appendix A.

3. Analysis of Quadratic Stability Conditions for Large Stiffness Values

We shall be content with these existence results on the feedback gains in the above analysis. However, let us note that if one takes the sufficient conditions for stability deduced from the above analysis, then the feedback gains λ_1 and $\lambda_2 \to +\infty$ as $k \to +\infty$. This suggests that in order to obtain quadratic Lyapunov stability of (1) one has to choose feedback gains proportional to the stiffness k as k becomes large. Let us rewrite (1) in state space form as

$$\begin{aligned} &z \in (\mathcal{NC}) \triangleq \{x : x < 0\} \quad \dot{z} = A_c z + \begin{pmatrix} 0 \\ \frac{k}{m} x \end{pmatrix} \\ &z \in (\mathcal{C}) \triangleq \{x : x \geq 0\} \qquad \dot{z} = A_c z \end{aligned} \tag{7}$$

where

$$z^T = \left(x - \frac{\lambda_1 x_d}{\lambda_1 + k}, \dot{x} \right) \tag{8}$$

$$A_c = \begin{pmatrix} 0 & 1 \\ \frac{-1}{m}(\lambda_1 + k) & \frac{-\lambda_2}{m} \end{pmatrix} \tag{9}$$

Clearly the choice of the first component of z stems from the fact that we want to stabilize the robot in contact with the environment. Moreover from (1) one sees that the equilibrium point of the first equation belongs to $(\mathcal{C})$, which means that the system in (7) possesses in fact only one equilibrium point, i.e. $z^T = (0, 0)$ (Note that the uniqueness holds for any value of x_d; when $x_d = 0$ both equations in (1) have the same equilibrium point $(x, \dot{x}) = (0, 0)$). Stability of A_c is independent of k since its eigenvalues are either real strictly negative or with real part equal to $\frac{-\lambda_2}{2m}$. Thus for any $Q_c = Q_c^T > 0$ there always exists $P = P^T > 0$ such that $A_c^T P + P A_c = -Q_c$. Since we want to stabilize the equilibrium point $z = 0$, we choose a Lyapunov function candidate as $V = z^T P z$. Along trajectories in $(\mathcal{NC})$ we get

$$\dot{V} = -z^T Q_c z + z^T P \begin{pmatrix} 0 \\ \frac{2k}{m} x \end{pmatrix} \tag{10}$$

For simplicity of the analysis, let us choose $x_d = 0$. Then we can write

$$\dot{V} = -z^T Q_c z + z^T P K z \triangleq -z^T \bar{Q}_c z \tag{11}$$

with

$$K \triangleq \begin{pmatrix} 0 & 0 \\ \frac{2k}{m} & 0 \end{pmatrix} \tag{12}$$

Simple calculations yield:

$$Q_c = \begin{bmatrix} 2\frac{\lambda_1 + k}{m} p_{12} & \frac{\lambda_2}{m} p_{12} + \frac{\lambda_1 + k}{m} p_{22} - p_{11} \\ \frac{\lambda_2}{m} p_{12} + \frac{\lambda_1 + k}{m} p_{22} - p_{11} & 2\left(\frac{\lambda_2}{m} p_{22} - p_{12}\right) \end{bmatrix} \tag{13}$$

$$Q_{nc} = \begin{bmatrix} \frac{2\lambda_1}{m} p_{12} & \frac{\lambda_2}{m} p_{12} + \frac{\lambda_1}{m} p_{22} - p_{11} \\ \frac{\lambda_2}{m} p_{12} + \frac{\lambda_1}{m} p_{22} - p_{11} & 2\left(\frac{\lambda_2}{m} p_{22} - p_{12}\right) \end{bmatrix} \tag{14}$$

where Q_{nc} is the symmetric part of the matrix $\bar{Q}_c$, that is independent of k. It is worth noting that only the skew-symmetric part of $\bar{Q}_c$ depends on k.
Thus a necessary and sufficient condition for Q_c to be positive definite is that:

- $\frac{\lambda_1+k}{m} p_{12} > 0$
- $$\begin{aligned} det(Q_c) = \; & 4\frac{\lambda_1+k}{m} p_{12}\left(\frac{\lambda_2}{m} p_{22} - p_{12}\right) \\ & - \left(\frac{\lambda_2}{m} p_{12} + \frac{\lambda_1+k}{m} p_{22} - p_{11}\right)^2 > 0 \end{aligned}$$

For Q_{nc} the conditions are the following :

- $\frac{2\lambda_1}{m} p_{12} > 0$
- $$\begin{aligned} det(Q_{nc}) = \; & 4\frac{\lambda_1}{m} p_{12}\left(\frac{\lambda_2}{m} p_{22} - p_{12}\right) \\ & - \left(\frac{\lambda_2}{m} p_{12} + \frac{\lambda_1}{m} p_{22} - p_{11}\right)^2 > 0 \end{aligned}$$

Our aim in this section is to examine the conditions such that this simple task is Lyapunov quadratically stable, and in particular to find out which kind of conditions this implies on the feedback gains. It is clear that if one concludes for instance that λ_1 must be larger than k, then it means that this sort of stability analysis is completely meaningless and useless as soon as the environment's stiffness is too large; then one has to change the objectives (relax the stability conditions) or the model (consider that the environment's surface is rigid) to get satisfactory conditions on the feedback gains. As shown in appendix B, the following result is true:

Claim 2. Consider the one-dof closed-loop equations in (1) with $x_d = 0$. Then quadratic stability of the system implies conditions such that when the environment's stiffness k grows unbounded, then the feedback gains λ_1 and/or λ_2 have to be chosen of order $\geq k^{\beta}$, $\beta \geq \frac{1}{2}$ to guarantee that the solution P of the Lyapunov equation remains bounded away from singularities (i.e. $\lambda_{\min} P \geq \delta > 0$ for some δ) and that the matrices Q_c and Q_{nc} remain positive definite.

We reiterate that the only thing we have done is to study conditions on the feedback gains such that the Lyapunov equation $A_c^T P + P A_c = -Q_c$ possesses a solution that is bounded-away from singularities and guarantees $Q_c > 0$, $Q_{nc} > 0$. The choice for such a stability analysis is quite natural: indeed it is the application of Lyapunov's direct method to a simple hybrid dynamical system [2]. The result of claim 2 are consistent with those to be found in other studies, see for instance [12] and [19] [18], although the system we analyze is much simpler that those treated in these references.

4. A Stiffness Independent Convergence Analysis

Firstly let us consider the system in (2). Let us take the Poincaré section $\Sigma^+ = \{(x, \dot{x})\colon x = 0, \dot{x}(t_k^+)\}$. Notice that if $x(0) > 0$, then the sequence of impact times $\{t_k\}$ is infinite (this can be easily shown by studying the vector field between the impacts, which forces the system to attain in finite time the constraint surface $x = 0$ whatever bounded initial conditions one may choose). The impact Poincaré map $P_\Sigma : \dot{x}(t_k^+) \to \dot{x}(t_{k+1}^+)$ is thus well defined. However it is not explicitely calculable, despite of the simplicity of the dynamics. This is due to the nonzero dissipation during flight-times. Let us choose:

$$V_\Sigma(k) = \frac{1}{2} m \dot{x}^2(t_k^+) \tag{15}$$

We prove that P_Σ is Lyapunov stable with V_Σ as a Lyapunov function as follows. Consider the function

$$V = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \lambda_1 (x - x_d)^2 \tag{16}$$

Along free-motion trajectories of (2) one obtains

$$\dot{V} = -\lambda_2 \dot{x}^2 \tag{17}$$

and at the impact times

$$\sigma_V(t_k) = \frac{1}{2} \left[\dot{x}^2(t_k^+) - \dot{x}^2(t_k^-)\right] = 0 \tag{18}$$

where $\sigma_f(t_k)$ generically denotes the jump in the function $f(\cdot)$ at t_k. Hence

$$V(t_{k+1}^+) - V(t_k^+) = -\lambda_2 \int_{(t_k, t_{k+1})} \dot{x}^2(\tau) d\tau \leq 0 \tag{19}$$

Now from the fact that $V(t_{k+1}^+) - V(t_k^+) = V_\Sigma(k+1) - V_\Sigma(k) \leq 0$, we conclude the proof. This stability result suggests [2] that one should be able to analyze the stability of the system in (1) for any $k \geq 0$, without the drawbacks encountered in the previous section.

In the following, we propose a different convergence analysis to prove that the equilibrium point of (1) is asymptotically reached for any initial condition and any value of the feedback gains, independently of the value of k; the particular feature of the analysis is that it extends naturally to the rigid environment case (i.e. $k = +\infty$), contrarily to the foregoing one. Roughly speaking, we consider a particular section of the phase-plane, $\Sigma = \{(x, \dot{x}) : x = 0\}$. Then we analyze the mass velocity at the instants t_i when the trajectories cross this section; we use the fact that these times define a sequence along which the kinetic energy is non-increasing; it follows that if $\{t_i\}$ is an infinite sequence, the velocity must converge to zero when $i \to +\infty$; if x_d is strictly positive, this leads to a contradiction and there is a finite number of bounces, so that both $\tilde{x}$ and $\dot{x}$ converge to zero. To clarify the notations the instants t_i and t_k are depicted in figure 2.

We assume that the mass makes contact with the environment at $t = t_i$, looses contact at $t = t_{i+1}$, $i \in \mathbb{N}$, and that contact occurs at $x = 0$. Thus contact occurs on intervals $[t_{2i}, t_{2i+1}]$, and free motion on intervals $[t_{2i+1}, t_{2i+2}]$. Let us consider the positive definite functions

$$V_c = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} (\lambda_1 + k) \left(x - \frac{\lambda_1 x_d}{\lambda_1 + k}\right)^2 \tag{20}$$

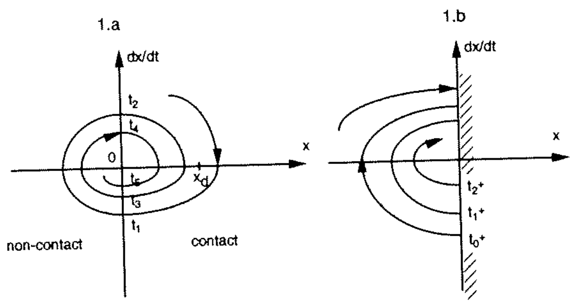

Figure 2. definition of the times t_i (1.a) and t_k (1.b).

and V_{nc} as in (16). On intervals $[t_{2i}, t_{2i+1}]$,

$$\dot{V}_c = -\lambda_2 \dot{x}^2 \tag{21}$$

On intervals $[t_{2i+1}, t_{2i+2}]$,

$$\dot{V}_{nc} = -\lambda_2 \dot{x}^2 \tag{22}$$

Let $T(t)$ denote the system's kinetic energy. From the fact that

$$V_c(t_{2i+1}) - V_c(t_{2i}) = T(t_{2i+1}) - T(t_{2i}) \tag{23}$$

and

$$V_{nc}(t_{2i+2}) - V_{nc}(t_{2i+1}) = T(t_{2i+2}) - T(t_{2i+1}) \tag{24}$$

we deduce that for all i, $T(t_{i+1}) - T(t_i) < 0$, hence

$$|\dot{x}(t_{i+1})| < |\dot{x}(t_i)| \tag{25}$$

The same inequalities hold for V_c and V_{nc}. Now notice that there are two situations: either the sequence of instants t_i is finite (the bounces stop after a finite time t_{2N} , $N < +\infty$, and since $x_d > 0$, $x(t) > 0$ for all $t > t_{2N}$), or this sequence is infinite i.e. $N = +\infty$.

• If $N < +\infty$, then for $t > t_{2N}$ the system is governed by the second equation in (1) (indeed each time the mass is "outside" the environment it necessarily collides again after a finite time) and we conclude that $x \to \frac{\lambda_1 x_d}{\lambda_1 + k}$, $\dot{x} \to 0$ asymptotically, globally and uniformly.

• Assume that $N = +\infty$. Since the kinetic energy is a positive definite function of the velocity that is non-increasing at times t_i, $T(t_i)$ converges as $i \to +\infty$, and so does $\dot{x}(t_i)$. Suppose that $|\dot{x}(t_i)| \to |\dot{x}_{ss}|$ with $|\dot{x}_{ss}| \geq \delta > 0$ (the ss subscript is for steady state value). Now $\delta > 0$ and since $sgn(\dot{x}(t_i)) = -sgn(\dot{x}(t_{i+1}))$ and $x(t_i) = x(t_{i+1}) = 0$, one deduces that the length of the orbit between t_i and t_{i+1} is strictly positive. Since the flow of both equations in (1) is exponential and bounded for bounded feedback gains and stiffness k, clearly $\mu[t_i, t_{i+1}] \triangleq \mu_{i+1} > 0$ and

$$T(t_i) - T(t_{i+1}) \triangleq \beta_{i+1} = \lambda_2 \int_{t_i}^{t_{i+1}} \dot{x}^2 dt > 0 \tag{26}$$

Note that for fixed and bounded coefficients in (1) μ_i and β_i depend only on δ (the other "initial" condition on the position needed to integrate the system on the interval $[t_i, t_{i+1}]$ remaining fixed at the times t_i) so that in particular $\beta_i \geq \beta(\delta) > 0$ for all $i \geq 0$ and $\delta > 0$. Since $T(t_i)$ is non-increasing, its limit value is its minimum value and for all $i \geq 0$, $|\dot{x}(t_i)| \geq |\dot{x}_{ss}| > \delta$. From the strictly positive variation of the kinetic energy we deduce that

$$\dot{x}^2(t_{i+1}) = \dot{x}^2(t_i) - \frac{2\beta_i}{m} \tag{27}$$

so that

$$\dot{x}_i^2 = \dot{x}_0^2 - \sum_{j=0}^{j=i-1} \beta_j \tag{28}$$

Therefore from the fact that the β_i's are strictly positive, we deduce that $|\dot{x}(t_i)|$ cannot converge towards a strictly positive $|\dot{x}_{ss}|$. Since however $T(t_i)$ and thus $\dot{x}(t_i)$ converge, we deduce that the only possible limit value for the velocity is $\dot{x}_{ss} = 0$. (Notice that if $\delta = 0$, then both μ_i and β_i may asymptotically take arbitrarily small values and $\dot{x}_i^2 = \dot{x}_0^2 - \sum_{j=0}^{j=i-1} \beta_j$ no longer leads to a contradiction). Thus we have shown that if there is an infinite number of bounces, then the value of the velocity when contact is established or lost ($x(t_i) = 0$) is bounded and tends to zero.

Having proved that the velocity $\dot{x}(t_i)$ tends to zero as $i \to +\infty$, we now show that the intervals Δ_i also converge to zero. Let us consider an arbitrarily large integer i such that $|\dot{x}(t_i)|$ is arbitrarily small, or in other words, for any $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that $i > N$ implies $|\dot{x}(t_i)| < \varepsilon$. We shall denote $\Delta_{i+1} \triangleq t_{i+1} - t_i$. First note that from any of the two dynamic equations in (1) we get $\Delta_i \leq \Delta_{\max} < +\infty$ for some $\Delta_{\max}$ since the "initial" velocities at times t_i are bounded and tend towards zero. Now we use the fact that both vector fields in (1) are explicitely integrable; assume that we place ourselves at t_{2i} such that $\dot{x}(t_{2i}) = \varepsilon > 0$, hence the system is in a contact phase for some time since $\ddot{x}(t_i) = \lambda_1 x_d - \lambda_2 \varepsilon > 0$ for some small enough $\varepsilon > 0$. We thus consider the second equation in (1). If the negative roots r_1 and r_2 of the characteristic equation are real and separated, i.e. $r_1 < r_2$, then the solution can be expressed as (recall that $x(t_i) = 0$ for all i):

$$x(t) = \gamma_1 e^{r_1(t-t_{2i})} + \gamma_2 e^{r_2(t-t_{2i})} + \bar{x}_d \tag{29}$$

with $\bar{x}_d = \frac{\lambda_1 x_d}{\lambda_1 + k}$, and $\gamma_1 = -\gamma_2 - \bar{x}_d$, $\gamma_1 r_1 = -\gamma_2 r_2 + \varepsilon$. Since we assume *a priori* that the sequence $\{t_i\}$ is infinite, t_{2i+1} exists and from (29) we get

$$\begin{aligned} \dot{x}(t_{2i+1}) &= \gamma_1 r_1 (e^{r_1 \Delta_{2i+1}} - (1-\varepsilon) e^{r_2 \Delta_{2i+1}}) \\ &= \gamma_1 r_1 e^{r_1 \Delta_{2i+1}} \left[1 - \left(1 - \frac{\varepsilon}{\gamma_1 r_1} \right) e^{(r_2 - r_1)\Delta_{2i+1}} + 1 \right] \end{aligned} \tag{30}$$

From the monotonicity of $\{|\dot{x}(t_i)|\}$ and its convergence, we deduce that $0 < |\dot{x}(t_{2i+1})| \le \varepsilon$. Assume now that the sequence $\{\Delta_{2i+1}\}$ does not converge towards zero, i.e. there exists $\Delta > 0$ such that $\Delta_{2i+1} \ge \Delta$ for all i. Then we get for any $\varepsilon > 0$, ε small enough:

$$\left| \left(1 - \frac{\varepsilon}{\gamma_1 r_1}\right) e^{(r_2 - r_1)\Delta_{2i+1}} - 1 \right| \ge \eta(r_1, r_2, \Delta) > 0 \tag{31}$$

This inequality is true because when $\varepsilon \to 0$ the left-hand-side of (31) tends to $|e^{(r_2-r_1)\Delta_{2i+1}} - 1|$ which cannot be zero since $r_1 < r_2$ and $\Delta_{2i+1} \ge \Delta > 0$. Furthermore $e^{r_1\Delta_{2i+1}} \ge \kappa(\Delta_{\max}, r_1) > 0$. Therefore from (30) we get:

$$|\gamma_1 r_1 \kappa(\Delta_{\max}, r_1)\eta(r_1, r_2, \Delta)| < \varepsilon \tag{32}$$

which cannot be true for ε small enough (Note that the roots as well as Δ and $\Delta_{\max}$ do not depend on ε). Since ε is strictly positive but arbitrarily small, we deduce that $\Delta_{2i+1} \to 0$ as $i \to +\infty$. A quite similar reasoning may be done for the case when $r_1 = r_2$.

When the roots are complex conjugate $r_1 = r + j\omega$, $r_2 = r - j\omega$, then the solution is given by:

$$x(t) = \gamma e^{r(t-t_{2i})}\cos(\omega(t - t_{2i}) + \varphi) + \bar{x}_d \tag{33}$$

with $\gamma = -\frac{\bar{x}_d}{\cos\varphi}$ and $\tan\varphi = \frac{\varepsilon + \bar{x}_d r}{\bar{x}_d \omega}$. Now we get

$$\dot{x}(t_{2i+1}) = \gamma e^{r\Delta_{2i+1}}\sqrt{r^2 + \omega^2}\cos(\omega\Delta_{2i+1} + \varphi + \Phi) \tag{34}$$

with $\tan\Phi = \frac{\omega}{r}$. Using the same arguments as in the real roots case, one sees that for $\dot{x}(t_{2i+1})$ to be arbitrarily small, we must have $\cos(\omega\Delta_{2i+1} + \varphi + \Phi)$ arbitrarily small, from which we deduce that $\omega\Delta_{2i+1} + \varphi + \Phi$ is arbitrarily close to $\frac{\pi}{2}$. Now for ε arbitrarily small, $\tan\varphi \to \frac{r}{\omega}$, and $\tan(\varphi + \Phi) \to +\infty$. But since Δ_{2i+1} is assumed to be bounded away from zero (and strictly positive by definition), $\tan(\frac{\pi}{2} - \omega\Delta_{2i+1})$ is clearly bounded. Thus by contradiction we deduce that $\{\Delta_{2i+1}\}$ converges to zero.

Now exactly the same reasoning may be done for the case of non-contact phases. It follows that if the velocities at times t_i converge towards zero, so do the intervals Δ_i. Since again the sequence $\{t_i\}$ is infinite, if its limit is infinite also then $(0, 0)$ is an equilibrium point of the system in (1). Clearly this is not the case, except if $x_d = 0$ (For the sake of briefness this case is not analyzed here; the analysis can be done using similar arguments).

In conclusion, we have proved that the sequence $\{t_i\}$ is either finite, or possesses a finite accumulation point. In both cases, we deduce that the equilibrium point of the system in (1) is asymptotically attained.

Relationship with the Case of a Rigid Environment

The only things that are modified in the rigid case are that since the intervals $[t_{2i}, t_{2i+1}] \to \{t_{2i}\}$, the distinction between instants t_{2i} and t_{2i+1} becomes worthless, and $\mu_{2i+1} = \beta_{2i+1} = 0$ while $\mu_{2i} > 0$ and $\beta_{2i} > 0$. One notes that in permanent contact (i.e. $x = \dot{x} \equiv 0$) and with $k = +\infty$, then

$$V_c = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}(\lambda_1 + k)x^2 - \lambda_1 x x_d + \frac{1}{2}\frac{\lambda_1^2 x_d}{\lambda_1 + k} \tag{35}$$

i.e.

$$V_c = \frac{1}{2}kx^2 = 0 \tag{36}$$

since the potential elastic energy vanishes as $k \to +\infty$ (indeed it can be verified that in this case, the roots r_1 and r_2 are necessarily complex conjugate, and that $x(t)$ varies proportionally to $\frac{1}{k}$). Moreover $\sigma_{V_c}(t_k) = \sigma_{V_{nc}}(t_k) = 0$. It follows that the contact phases reduce to the impact times t_k. One retrieves the analysis done at the beginning of this section by studying the variation of V_{nc} between the impacts (see (17)) and at the impact times (see (18)). Therefore the stability analysis for the compliant case $k < +\infty$ naturally reduces to the stability analysis of the impact Poincaré map in the rigid case. In addition we have proved asymptotic stability.

We thus have proved the following:

Claim 3. Consider the closed-loop equations in (1). Then for any $\lambda_1 > 0$, $\lambda_2 > 0$, $k \in [0, +\infty]$, and for all initial conditions $x(0)$, $\dot{x}(0)$, $x \to \frac{\lambda_1 x_d}{\lambda_1 + k}$ and $\dot{x} \to 0$ as $t \to +\infty$.

Remark 1. A distinction has to be made between two different cases of analysis: We may consider i) Either an arbitrarily large but bounded k, ii) or a k that tends to infinity (that is implicitely a sequence of stiffnesses k_n with unbounded limit together with the corresponding dynamics). Clearly claim 3 can be concluded from the analysis in sections 2.2 and 3 in case i), but not in case ii). The utility of the analysis proposed in this section is to enable us to draw conclusions in both the compliant and the rigid environment cases with a unique philosophy of stability analysis.

Remark 2. Since the solutions of (1) converge to those of (2), it would be interesting to reverse the reasoning done in section 4 as follows: if the fixed point of (2) is stable, then the fixed point of (1) is stable also for a large enough k. One could think of first studying the stability of the rigid body system impact Poincaré map, and then draw conclusions on the stability of an approximating compliant problem. This of course relies on the ability of proving the closeness of solutions of both systems for large enough k. Moreover it may be difficult to study the Poincaré map stability.

Remark 3. It is not realistic in general to assume that the interaction force F is measurable and can be compensated for, during the rebounds phase. Indeed if k is large, then F will be large also and its support (as a time function) very small (F approaches a Dirac distribution). This is a motivation to search for feedback control algorithms that are able to stabilize the transition phases without force feedback, see e.g. [19]. In fact it has been recognized [21] that in general 3 distinct controllers have to be used for the control of a *complete* robotic task (i.e. a task involving free-motion as well as constrained motion phases, and transition phases). The algorithm analyzed in this paper could be used in this setting as a transition phase controller, as part of a more sophisticated controller (in the spirit of what has been proposed e.g. in [12] [3]). This could also be the starting point to a unified stability analysis framework encompassing both compliant and rigid body based models.

5. Conclusions

In this note, we have studied the stability of a simple robotic task that includes both contact and non-contact phases, when a PD motion controller is applied. It may also be considered as the stability analysis of a contact phase that takes into account the unilateral feature of the constraint surface. An interesting problem is to study conditions that guarantee that if the robot's tip happens to take off the surface (either because it has been initialized outside the environment or because of an oscillatory behaviour or due to a disturbance) then it will eventually attain its steady state value after a possible sequence of rebounds. We have studied what happens when the environment's stiffness grows very large and when one desires quadratic stability of the closed-loop system with well-conditioned solutions P to the Lyapunov equation, i.e. when the minimum eigenvalue of P is required to remain larger than some strictly positive constant. It appears that the velocity feedback gain must be chosen proportional to the environment's stiffness k, so that these sufficient conditions are clearly useless for practical purposes as soon as the contact stiffness is too large. The results indicate that the conditions that one may derive from an analysis done with continuous dynamic models may not be feasible for the environments which are too rigid; although we do not claim that such conclusions hold for *any* control algorithm for robot manipulators with *any* compliant environment, we note that this is quite consistent with the sufficient conditions found in the literature on the subject. Finally we have shown that one can conclude on global asymptotic convergence of the tracking errors towards zero independently of the values of the feedback gains and of the environment's stiffness. This is in accordance with the results obtained for the impact Poincaré map in the case of a rigid body model. We have proposed a convergence proof that encompasses both compliant and rigid cases, a property that is clearly not shared by some the other classical stability analysis.

A. Proof of Claim 1

λ_1 and λ_2 can be chosen such that $a_{nc} > 0$ and $b_{nc} > 0$. Thus we conclude that for all x:

$$\dot{V} \leq -\alpha \dot{x}^2 - \beta \tilde{x}^2 + R \tag{37}$$

with $\alpha = \min(a_{nc}, a_c)$, $\beta = \min(b_{nc}, b_c)$. Following the arguments in [6], we deduce that the state $(\tilde{x}, \dot{x})$ converges in finite time in a ball with radius r, with $r \to 0$ as λ_1 and λ_2 tend to $+\infty$. Therefore for all $t \geq \bar{t}$, $\bar{t} < +\infty$, we get $|\tilde{x}| < r$. Now notice (see (4)) that as $\lambda_1 \to +\infty$ then:

$$R \to \frac{1}{2} k^2 x_d^2 \left(1 + \frac{c^2}{m^2}\right) \tag{38}$$

Since by taking λ_1 and λ_2 large enough r can be made arbitrarily small and since $\frac{\lambda_1 x_d}{\lambda_1+k} \to x_d$ as $\lambda_1 \to +\infty$, it follows that for λ_1 and λ_2 large enough, then $|\tilde{x}(t)| > r$ for $t \geq \bar{t}$ implies that $x(t) > 0$ for all $t \geq \bar{t}$. Then (6) implies that both $\dot{x}$ and $\tilde{x}$ converge asymptotically to zero. Notice that outside some ball $B_{\bar{R}}$ we have for some $Q = Q^T > 0$:

$$\dot{V} \leq -z^T Q z \tag{39}$$

with $z^T = (\tilde{x}, \dot{x})$. This can be easily deduced by spliting α and β into α_1 and α_2, β_1 and β_2. Then one obtains:

$$\dot{V} \leq -\alpha_1 \dot{x}^2 - \beta_1 \tilde{x}^2 - \alpha_2 \dot{x}^2 - \beta_2 \tilde{x}^2 + R \tag{40}$$

so that outside the ball $B_{\bar{R}}$ with $\bar{R} = \frac{R}{\min(\alpha_2, \beta_2)}$ we get:

$$\dot{V} \leq -\alpha_1 \dot{x}^2 - \beta_1 \tilde{x}^2 \tag{41}$$

Still λ_1 and λ_2 can be chosen large enough so that $\bar{R}$ is as small as desired. Thus we deduce that for $0 \leq t \leq \bar{t}$:

$$||z(t)|| \leq \sqrt{\frac{V(0)}{\lambda_{\min} P}} \exp\left(-\frac{\lambda_{\min} Q}{\lambda_{\max} P} t\right) \tag{42}$$

i.e. the ball $B_{\bar{R}}$ is reached exponentially fast.

B. Proof of Claim 2

Starting from the Lyapunov equation, one may first fix Q_c as a positive definite matrix and then try to calculate the unique corresponding positive definite P (see e.g. [20] lemma 42, chapter 5). A second way to attack the problem is to pick a $P > 0$ and study the properties of the resulting Q_c ([20] p.198). Instead of choosing a $Q_c > 0$ and solving the Lyapunov equation for P, we rather consider a matrix P and find conditions such that the corresponding Q_c is positive definite, together with Q_{nc}. Thus we prove that the only way for P not to tend towards a singular matrix while keeping $Q_c > 0$ and $Q_{nc} > 0$ when k increases is to take the gain λ_1 of the same order as k^2.

The above determinants can be written in the following way:

$$\begin{aligned}
det(Q_{nc}) &= 4\frac{\lambda_1}{m}(p_{11}p_{22} - p_{12}^2) - \left(\frac{\lambda_1}{m}p_{22} + p_{11} - \frac{\lambda_2}{m}p_{12}\right)^2 \\
det(Q_c) &= 4\frac{\lambda_1 + k}{m}(p_{11}p_{22} - p_{12}^2) - \left(\frac{\lambda_1 + k}{m}p_{22} + p_{11} - \frac{\lambda_2}{m}p_{12}\right)^2 \\
&= -\frac{1}{m^2}(\lambda_1 p_{22} + m p_{11} - \lambda_2 p_{12})^2 - 2\frac{k p_{22}}{m^2}(\lambda_1 p_{22} + m p_{11} - \lambda_2 p_{12}) \\
&\quad + 4\frac{\lambda_1 + k}{m}(p_{11}p_{22} - p_{12}^2) - \left(\frac{k p_{22}}{m}\right)^2
\end{aligned}$$

Let us denote $Y \triangleq \lambda_1 p_{22} + mp_{11} - \lambda_2 p_{12}$ and $|P| = p_{11}p_{22} - p_{12}^2$ then:

- $det(Q_{nc}) > 0 \Leftrightarrow 4\lambda_1 m|P| - Y^2 > 0$
- $det(Q_c) > 0 \Leftrightarrow Y^2 + 2kp_{22}Y - 4(\lambda_1 + k)m|P| + (kp_{22})^2 < 0$

We deduce that Y has to satisfy the following inequalities:

$$\begin{aligned} &-2\sqrt{m\lambda_1|P|} < Y < 2\sqrt{m\lambda_1|P|} \\ &-k\,p_{22} - 2\sqrt{m(\lambda_1 + k)|P|} < Y < -k\,p_{22} + 2\sqrt{m(\lambda_1 + k)|P|} \end{aligned} \tag{43}$$

Since $-k\,p_{22} - 2\sqrt{m(\lambda_1 + k)|P|} < -2\sqrt{m\lambda_1|P|}$ there exists a solution for Y if and only if $-2\sqrt{m\lambda_1|P|} < -k\,p_{22} + 2\sqrt{m(\lambda_1 + k)|P|}$, which is found after some manipulations to be equivalent to the following conditions:

$$\begin{cases} 2\Lambda_k^2\,p_{11} - 2\sqrt{\Lambda_k^2(\Lambda_k^2\,p_{11}^2 - p_{12}^2)} < p_{22} < 2\Lambda_k^2\,p_{11} + 2\sqrt{\Lambda_k^2(\Lambda_k^2\,p_{11}^2 - p_{12}^2)} \\ p_{12} < \Lambda_k\,p_{11} \end{cases} \tag{44}$$

with $\Lambda_k = \frac{\sqrt{m(\lambda_1+k)}+\sqrt{m\lambda_1}}{k}$.

Notice that by choosing $p_{22} = 2\Lambda_k^2 p_{11}$ we can find P that satisfies (44) and that is positive-definite.

From (43), Y satisfies the following inequalities:

$$-2\sqrt{m\lambda_1|P|} < Y < \min(-k\,p_{22} + 2\sqrt{m(\lambda_1 + k)|P|}, 2\sqrt{m\lambda_1|P|}) \tag{45}$$

It is easy to prove that $\lambda_{\min}(P) \leq p_{22}$ and $\lambda_{\max}(P) \geq p_{11}$ from which we deduce that

$$\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \geq \frac{p_{11}}{p_{22}} \tag{46}$$

From (44), we can write $p_{22} < 4\Lambda_k^2\,p_{11}$, thus P has bounded entries when p_{11} is bounded and the above conditions fulfilled. Then if p_{11} is a finite real number the conditions of existence of Y imply that the coefficients p_{12} and p_{22} tend to zero when the stiffness of the environment becomes infinite, rendering the matrix P singular. Let us note that the stability analysis then becomes asymptotically (i.e. when $k \to +\infty$) meaningless since Q_{nc} in (14) has bounded entries. The only way to avoid this problem is to increase the gain λ_1 such that the coefficient Λ_k does not tend towards zero when the stiffness increases, i.e. λ_1 has to be chosen of order $\geq k^2$. Assume that this is done so that P is well conditioned, and let us examine how λ_2 has to be chosen. λ_2 may be found by using (45):

$$\frac{mp_{11} + \lambda_1 p_{22} - Y_{\max}}{p_{12}} < \lambda_2 < \frac{mp_{11} + \lambda_1 p_{22} + 2\sqrt{m\lambda_1|P|}}{p_{12}} \tag{47}$$

where $Y_{\max} = \min(-k\,p_{22} + 2\sqrt{m(\lambda_1 + k)|P|}, 2\sqrt{m\lambda_1|P|})$. This implies that when λ_1 is of order k^2 and k grows unbounded, the gain λ_2 becomes infinite too.

Let us examine what happens if we allow p_{11} to be proportional to k^α, $\alpha > 1$. Then p_{22} may be chosen of order $\leq k^{\alpha-1}$ from (44). Also p_{12} will be of order $\leq k^{\alpha-\frac{1}{2}}$ from the second condition in (44). Now from (47) we have the following: If $Y_{\max} \leq 0$ then obviously λ_2 is of order $k^{\frac{1}{2}}$ as $k \to +\infty$. If $Y_{\max} > 0$, let us analyze the case when $Y_{\max} = 2\sqrt{m\lambda_1|P|}$: This value is maximum when p_{12} is minimum, i.e. bounded, and when both p_{11} and p_{22} are maximum, i.e. respectively of orders k^{α} and $k^{\alpha-1}$; then $Y_{\max}$ is of order $k^{\alpha-\frac{1}{2}}$ so that λ_2 grows as $k^{\frac{1}{2}}$. Now if $Y_{\max} \triangleq A = -k\,p_{22} + 2\sqrt{m(\lambda_1 + k)|P|}$ that we assume > 0: then necessarily since $p_{22} \geq 0$, the second term in A is at least of the same order as kp_{22} in k as k grows unbounded. Thus at most the order of $Y_{\max}$ will be that of the second term $2\sqrt{m(\lambda_1 + k)|P|}$, which is found to be at most k^{α}. But if this is the case then this term dominates $2\sqrt{m\lambda_1|P|}$ and asymptotically (in k) $Y_{\max}$ will necessarily be equal to this last term, hence we are back to the previous case. Now if the order of A is k^{γ} with $\gamma < \alpha$ then p_{11} will asymptotically dominate $Y_{\max}$ and the left-hand-side of (47) is asymptotically of order $k^{\frac{1}{2}}$. Thus λ_1 may be chosen bounded but λ_2 will grow unbounded to guarantee $\lambda_{\min}P \geq \delta > 0$ for any arbitrarily small but fixed δ and $Q_c > 0$, $Q_{nc} > 0$.

C. Impulsive Dynamics

For the sake of briefness, we shall not recall here all the details about impulsive impact dynamics. Let us simply recall some basic facts: The interaction forces between two rigid bodies are mathematically modelled by Dirac distributions $p\delta_t$, whose magnitude p may be calculated from the velocities before the impact and using a physical law of percussion like Newton's restitution coefficient $0 \leq e \leq 1$. $e = 1$ corresponds to the case when there is no loss of energy at the impact, so that in our case both velocities of the mass after and before the impact are equal in magnitude (and of course of opposite signs). In (2) we get $p(t_k) = -2m\dot{x}(t_k^-)$. Equations containing Dirac measures are called *measure differential equations*; the solutions of such equations are right-continuous functions of *local bounded variation* [16]. Stability in the sense of Lyapunov and Lyapunov functions can still be considered for measure differential equations [9][3]: One has however to consider in place of the traditional derivative $\dot{V}$ of a positive definite function V its right Dini derivative[4] in intervals between impacts (smooth dynamics), and its jumps $\sigma_V(t_k) \triangleq V(t_k^+) - V(t_k^-)$ at instants of discontinuities t_k, $i \geq 0$. The latter may also be seen as an application of a generalized chain rule for distributional derivatives to the function $V(x(t), \dot{x}(t))$ where $\dot{x}(t)$ has local bounded variation and $V(\cdot,\cdot)$ is continuously differentiable: Then at $t = t_k$, $DV = \sigma_V(t_k)\delta_{t_k}$ (D usually denotes distributional derivatives of functions of bounded variation, see e.g. [16]). Rigorous convergence of sequences of continuous-dynamics problems (or "approximating problems") towards a nonsmooth dynamical problem when $k \to +\infty$ has been treated for instance in [15].

Notes

1. For instance [25] show that an integral force feedback helps in stabilizing the impact phase when the environment is (sufficiently) rigid, whereas [21] show that it is not suitable for a (sufficiently) flexible environment.
2. Notice that we have not proved the *asymptotic* stability of P_{Σ}.

3. Let us note however that the results in [9] do not apply here since the author studies stability of systems of the form $\dot{x} = f(t, x) + Du$, u of local bounded variation, and Du is to be considered as a disturbance on $\dot{x} = f(t, x)$ with globally asymptotically stable equilibrium point $x = 0$; In our case the impacts will drive the system to another point than the equilibrium point of the smooth dynamics (that may even possess no equilibrium point: The reader may think of the bouncing ball problem to illustrate clearly this). More details can be found in [27] chapter 7.
4. $\dot{V}(t) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t + h) - V(t)]$.

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