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Mathematics of Control, Signals, and Systems

The Krakovskii-LaSalle Invariance Principle for a Class of Unilateral Dynamical Systems*

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Abstract. This paper is devoted to the study of the extension of the invariance lemma to a class of hybrid dynamical systems, namely evolution variational inequalities. Applications can be found in models of electrical circuits with ideal diodes or oligopolistic market equilibrium.

Key words. Lyapunov stability, Unilateral constraints, LaSalle invariance principle, Variational inequalities, Convex analysis, Hybrid dynamics.

1. Introduction

Recent research in systems and control saw an unprecedented rise in the analysis and control of hybrid dynamical systems [1]–[5]. Stability issues are of utmost importance for this class of complex systems [9]–[12], [15], [18], [19], [27]–[30], [35]. Due to the high complexity and intricacy of the dynamics of hybrid systems, it is almost necessary to focus on specific subclasses. Complementarity systems constitute a subclass which has a particular interest since it has a strong structure and is at the same time rich in terms of potential applications [12]–[14], [18]–[22]. Roughly speaking, complementarity dynamical systems are made of a continuous dynamical system, coupled to a set of algebraic relations that involve complementarity conditions between two variables, one of which is a Lagrange multiplier. As we shall briefly see below, such systems are related to evolution variational inequalities and to differential inclusions. They can also be seen as a sort of nonsmooth differential-algebraic system [12]. All these different formalisms are a consequence of convex analysis and complementarity theory. It is the framework of evolution variational inequalities that will be chosen here.

The extension of Lyapunov stability and of Lyapunov's second method for hybrid systems has been studied by various authors. As already known, the stability of

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unilaterally constrained systems can differ a lot from that of unconstrained systems, see e.g., [35, example 3.2], [9]. Therefore the development of suitable stability tools is necessary for such nonsmooth systems. Since it is, in general, difficult to get a negative definite derivative of a considered Lyapunov function, the Krakovskii–LaSalle invariance lemma is a widely used result for asymptotic stability [25], [33], [34]. It is well known that unilaterally constrained systems may have solutions which are discontinuous in the initial conditions [20], [24]. Consequently, the extension of the invariance lemma to hybrid systems requires attention [29], [31], [32], and the same applies to the systems considered in this paper.

Let us deal with the following class of dynamical systems:

Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a nonlinear operator. For $(t_0, x_0) \in \mathbb{R} \times K$, we consider the problem $P(t_0, x_0)$: Find a function $t \to x(t)$ $(t \geq t_0)$ with $x \in C^0([t_0, +\infty); \mathbb{R}^n)$, $(dx/dt) \in L^\infty_{\text{loc}}(t_0, +\infty; \mathbb{R}^n)$ and such that:

$$\begin{cases} x(t) \in K, \ t \geq t_0, \\ \langle \frac{dx}{dt}(t) + F(x(t)), v - x(t) \rangle \geq 0, \quad \forall v \in K, \quad \text{a.e. } t \geq t_0, \\ x(t_0) = x_0. \end{cases} \tag{1}$$

Here $\langle ., . \rangle$ denotes the euclidean scalar product in $\mathbb{R}^n$. The system in (1) is an evolution variational inequality which we denote as $\text{LEVI}(A, K)$ when $F \equiv A$, with $A \in \mathbb{R}^{n \times n}$ denoting a real matrix. It follows from standard convex analysis that (1) can be rewritten equivalently as the differential inclusion

$$\begin{cases} \frac{dx}{dt}(t) + F(x(t)) \in -N_K(x(t)), \\ x(\cdot) \in K, \end{cases} \tag{2}$$

where $N_K(x(t)) = \{s \in \mathbb{R}^n : \langle s, v - x(t) \rangle \leq 0, \ \forall \ v \in K\}$ is the normal cone to K at $x(t)$ [23]. In case $K = \{x \in \mathbb{R}^n : Cx + d \geq 0\}$ for some matrix $C \in \mathbb{R}^{m \times n}$ and vector $d \in \mathbb{R}^m$, we can rewrite (1) as

$$\begin{cases} \frac{dx}{dt}(t) + F(x(t)) = C^{\mathrm{T}}\lambda, \\ 0 \leq y = Cx + d \perp \lambda \geq 0, \end{cases} \tag{3}$$

where $\lambda \in \mathbb{R}^m$ is a Lagrange multiplier, and the second line of (3) means that both y and λ have to be nonnegative and orthogonal. These conditions are called *complementarity conditions*. Let us note that such models are used to represent the dynamics of various systems like oligopolistic markets, spatial price, elastic demand traffic [35], and some electrical circuits with ideal diodes [19], [10].

Remark 1. It is noteworthy that the boundary of the set K may possess an infinity of corners. Consequently, it may not be suitable for a description of (1) as in (3). The results of this paper continue to hold even in this case. More generally, the relationships between various formalisms which are used in the literature (variational inequalities [15], Filippov's differential inclusions [29], Moreau's sweeping process [37], complementarity systems [19], [20], hybrid systems [32], piecewise-linear systems [31], etc.) need to be investigated further. However, this is outside the topic of this paper.

Remark 2. The convention adopted in (1) to write the vector field $F(x)$ is not the usual one in systems and control theory, but is the standard one in variational inequality theory [8], [7], [15]. It is adopted in this paper. Consequently, the sign conditions on the Lyapunov functions derivatives, will be reversed compared to the usual convention.

The paper is organised as follows: in Section 2, some results on existence, uniqueness, continuous dependance, and stability of solutions are given. Section 3 is devoted to prove the invariance theorem in a general setting. Section 4 applies the previous results to linear evolution variational inequalities, and section 5 deals with nonlinear evolution variational inequalities. In Section 6, an example is presented, and conclusions end the paper in Section 7.

Notations: $f'(x)$ is the gradient of the function $f(\cdot)$ at x, $\bar{S}$ denotes the closure of the set S and $W^{1,1}(I, \mathbb{R}^n)$ denotes the Sobolev space of L^1−functions defined on I with argument in $\mathbb{R}^n$.

2. Background and Preliminaries

Let us first specify some conditions ensuring the existence and uniqueness of the initial value problem $P(t_0, x_0)$. The following existence and uniqueness result is a direct consequence of [15, corollary 2.2].

Assumption 1. *K is a nonempty closed convex subset. The operator $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous operator such that for some $\omega \geq 0$, $F + \omega I$ is monotone.*

Theorem 1. *Let assumption 1 be satisfied. Let $t_0 \in \mathbb{R}$ and $x_0 \in K$ be given. Then, there exists a unique $x \in C^0([t_0, +\infty); \mathbb{R}^n)$ such that*

$$\frac{dx}{dt} \in L^\infty_{\text{loc}}(t_0, +\infty; \mathbb{R}^n), \tag{4}$$

$$x \text{ is right-differentiable on } [t_0, +\infty), \tag{5}$$

$$x(t_0) = x_0, \tag{6}$$

$$x(t) \in K, \ t \geq t_0, \tag{7}$$

$$\langle \frac{dx}{dt}(t) + F(x(t)), v - x(t) \rangle \geq 0, \quad \forall v \in K, \ \text{a.e. } t \geq t_0. \tag{8}$$

Remark 3. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ can be written as

$$F(x) = Ax + \Phi'(x) + F_1(x), \ \forall x \in \mathbb{R}^n,$$

where $A \in \mathbb{R}^{n\times n}$ is a real matrix, $\Phi \in C^1(\mathbb{R}^n; \mathbb{R})$ is convex and F_1 is Lipschitz continuous, i.e.,

$$\|F_1(x) - F_1(y)\| \leq k\|x - y\|, \forall x, y \in \mathbb{R}^n,$$

for some constant $k > 0$. Then, F is continuous and $F + \omega I$ is monotone provided that $\omega \geq 0$ is chosen great enough, i.e.

$$\omega \geq \sup_{\|x\|=1} \langle -Ax, x \rangle + k.$$

For instance $F(x) = -2x + 4x^3 + \cos(x)$ satisfies the requirements with $A = -2$, $\Phi(x) = x^4$, $F_1(\cdot) = \cos(\cdot)$.

Let us recall that an operator $F(\cdot)$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ is monotone if $\langle w - w', x - x' \rangle \geq 0$ for all couples (w, x) and (w', x') that belong to its graph. Suppose that the assumption 1 is satisfied and denote by $x(.; t_0, x_0)$ the unique solution of Problem $P(t_0, x_0)$. We prove below that for $t \geq t_0$ fixed, the function $x(t; t_0, .)$ is continuous on K.

Theorem 2 (Continuity in the initial conditions). *Suppose that the assumption 1 holds. Let $t \geq t_0$ be fixed. The function*

$$x(t; t_0, .) : K \to \mathbb{R}^n; x_0 \to x(t; t_0, x_0)$$

is continuous.

Proof. Let $x_0 \in K$ be given and let $\{x_{0,i}\} \subset K$ such that $x_{0,i} \to x_0$ in $\mathbb{R}^n$. Let us here set $x(t) := x(t; t_0, x_0)$ and $x_i(t) := x(t; t_0, x_{0,i})$. We know that

$$\langle \frac{dx}{dt}(t) + F(x(t)), v - x(t) \rangle \geq 0, \quad \forall v \in K, \ \text{ a.e. } t \geq t_0 \tag{9}$$

and

$$\langle \frac{dx_i}{dt}(t) + F(x_i(t)), z - x_i(t) \rangle \geq 0, \quad \forall z \in K, \ \text{ a.e. } t \geq t_0. \tag{10}$$

Setting $v = x_i(t)$ in (9) and $z = x(t)$ in (10), we obtain the relations:

$$-\langle \frac{dx}{dt}(t) + F(x(t)), x_i(t) - x(t) \rangle \leq 0, \quad \text{a.e. } t \geq t_0 \tag{11}$$

and

$$\langle \frac{dx_i}{dt}(t) + F(x_i(t)), x_i(t) - x(t) \rangle \leq 0, \quad \text{a.e. } t \geq t_0. \tag{12}$$

It results that

$$\begin{aligned} \langle \frac{d}{dt}(x_i - x)(t), x_i(t) - x(t) \rangle &\leq \omega \|x_i(t) - x(t)\|^2 \\ &\quad - \langle [F + \omega I](x_i(t)) - [F + \omega I](x(t)), x_i(t) - x(t) \rangle, \ \text{ a.e. } t \geq t_0. \end{aligned}$$

Our hypothesis ensure that $F + \omega I$ is monotone. It results that

$$\frac{d}{dt}\|x_i(t) - x(t)\|^2 \leq 2\omega \|x_i(t) - x(t)\|^2, \quad \text{a.e. } t \geq t_0. \tag{13}$$

Using some Gronwall inequality (see e.g., Lemma 4.1 in [6]), we get

$$\|x_i(t) - x(t)\|^2 \leq \|x_{0,i} - x_0\|^2 e^{2\omega(t - t_0)}, \quad \forall t \geq t_0. \tag{14}$$

It follows that for $t \geq t_0$ fixed, $x(t; t_0, x_{0,i}) \to x(t; t_0, x_0)$ as $i \to \infty$. ∎

A similar result has been presented in [35, Theorem 2.9] in the framework of projected dynamical systems. However, the proof of Theorem 2 differs considerably. Suppose now in addition to assumption 1 that $0 \in K$ and the following holds

Assumption 2.

$$\langle F(0), h\rangle \geq 0, \quad \forall h \in K. \tag{15}$$

Then

$$x(t; t_0, 0) = 0, \quad \forall t \geq t_0,$$

i.e., the trivial stationary solution 0 is the unique solution of problem $P(t_0, 0)$.

We may now define the stability of the trivial solution as in [15]. The stationary solution 0 is called stable if small perturbations of the initial condition $x(t_0) = 0$ lead to solutions which remain in the neighborhood of 0 for all $t \geq t_0$, precisely:

Definition 1. The equilibrium point $x = 0$ is said to be stable in the sense of Lyapunov if for every $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon) > 0$ such that for any $x_0 \in K$ with $\|x_0\| \leq \eta$, the solution $x(\cdot; t_0, x_0)$ of problem $P(t_0, x_0)$ satisfies $\|x(t; t_0, x_0)\| \leq \varepsilon,\ \forall t \geq t_0$.

If in addition, the trajectories of the perturbed solutions are attracted by 0, then we say that the stationary solution is asymptotically stable, precisely:

Definition 2. The equilibrium point $x = 0$ is asymptotically stable if it is stable, and there exists $\delta > 0$ such that for any $x_0 \in K$ with $\|x_0\| \leq \delta$, the solution $x(\cdot; t_0, x_0)$ of problem $P(t_0, x_0)$ fulfills

$$\lim_{t\to+\infty} \|x(t; t_0, x_0)\| = 0.$$

Let us now recall general abstract theorems of stability and asymptotic stability in terms of generalized Lyapunov functions $V \in C^1(\mathbb{R}^n; \mathbb{R})$. The following results are particular cases of those proved in [15].

Theorem 3. *Suppose that the assumptions 1 and 2 hold. Suppose that there exists* $\sigma > 0$ *and* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1)
$$V(x) \geq a(\|x\|),\ x \in K,\ \|x\| \leq \sigma,$$
with $a : [0, \sigma] \to \mathbb{R}$ *satisfying* $a(t) > 0,\ \forall t \in (0, \sigma)$; *[resp.* $a(t) \geq ct^\tau$, $\forall t \in [0, \sigma]$, *for some constants* $c > 0,\ \tau > 0$*];*

(2) $V(0) = 0$;

(3) $x - V'(x) \in K,\quad$ *for all* $x \in \partial K,\ \|x\| \leq \sigma$*;*

(4) $\langle F(x), V'(x)\rangle \geq 0,\ [\langle F(x), V'(x)\rangle \geq \lambda V(x)]$ *for all* $x \in K,\ \|x\| \leq \sigma$.

Then, the trivial solution of (7)–(8) is stable [resp. asymptotically stable]. ■

Assumption (3) implies that $-V'(x) \in T_K(x)$ for all $x \in \partial K$, $||x|| \leq \sigma$, where $T_K(x)$ is the tangent cone to K at x [23, Prop. 5.2.1]. Assumptions (3) and (4) are illustrated on Fig. 1, where the ellipsoid stands for some level set of the Lyapunov function.

Let us here denote by $\mathcal{S}$ the set of stationary solutions of (7)–(8), that is,

$$\mathcal{S} := \{z \in K : \langle F(z), v - z\rangle \geq 0, \forall v \in K\}.$$

Condition (15) ensures that $0 \in \mathcal{S}$.

Let us now remark that some conditions for asymptotic stability invoked in Theorem 3 ensure that the trivial stationary solution of (7)–(8) is isolated in $\mathcal{S}$.

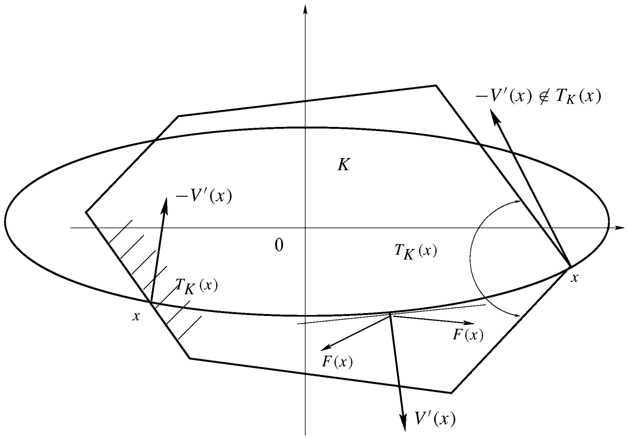

Fig. 1. Conditions on the Lyapunov function.

Proposition 1. *Suppose that the assumptions 1 and 2 hold. Suppose that there exists* $\sigma > 0$ *and* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1) $x - V'(x) \in K$, *for all* $x \in \partial K$, $\|x\| \leq \sigma$;
(2) $\langle F(x), V'(x)\rangle \geq 0$, *for all* $x \in K$, $\|x\| \leq \sigma$;
(3) $E := \{x \in K, \|x\| \leq \sigma : \langle F(x), V'(x)\rangle = 0\} = \{0\}$.

Then, the trivial stationary solution of (7)–(8) is isolated in $\mathcal{S}$. *If the same conditions hold with* $\sigma = +\infty$, *then* $\mathcal{S} = \{0\}$, *i.e., the trivial stationary solution of (7)–(8) is the unique stationary solution of (7)–(8).*

Proof. Let $0 < \delta \leq \sigma$ be given and let $B_\delta := \{x \in \mathbb{R}^n : \|x\| \leq \delta\}$. We claim that $B_\delta \cap \mathcal{S} = \{0\}$. Indeed, let $z \in B_\delta \cap \mathcal{S}$ be given. We have $z \in K$ and

$$\langle F(z), v - z\rangle \geq 0, \quad \forall v \in K. \tag{16}$$

We claim that

$$\langle F(z), V'(z)\rangle \leq 0.$$

Indeed, if $z \in \partial K$, then $z - V'(z) \in K$ and thus setting $v = z - V'(z)$ in (16), we get $\langle F(z), V'(z)\rangle \leq 0$. If $z \in \text{int}\{K\}$, then there exists $\varepsilon > 0$ such that $z - \varepsilon V'(z) \in K$, and thus setting $v = z - \varepsilon V'(z)$ in (16), we obtain $\varepsilon\langle F(z), V'(z)\rangle \leq 0$. Thus, $\langle F(z), V'(z)\rangle \leq 0$ since $\varepsilon > 0$.
Now using assumption (2), we obtain

$$\langle F(z), V'(z)\rangle = 0.$$

Finally, assumption (3) yields $z = 0$. The last part of the proposition can be proved by following the same argument. ■

Note that condition (4) for asymptotic stability in Theorem 3 yields conditions (2) and (3) in Proposition 1. It results that the assumptions for asymptotic stability done in Theorem 3 ensure that the trivial solution of (7)–(8) is isolated in $\mathcal{S}$.

3. The Invariance Theorem

Suppose that assumption 1 holds. For $x_0 \in K$, we denote by $\gamma(x_0)$ the *orbit*

$$\gamma(x_0) := \{x(\tau; t_0, x_0); \tau \geq t_0\}$$

and by $\Lambda(x_0)$ the *limit set*

$$\Lambda(x_0) := \{z \in \mathbb{R}^n : \exists\{\tau_i\} \subset [t_0, +\infty); \tau_i \to +\infty \text{ and } x(\tau_i; t_0, x_0) \to z\}.$$

We say that a set $\mathcal{D} \subset K$ is *invariant* provided that

$$x_0 \in \mathcal{D} \Rightarrow \gamma(x_0) \subset \mathcal{D}.$$

Theorem 4 (Invariance Theorem). *Suppose that the assumption 1 holds. Let $\Psi \subset \mathbb{R}^n$ be a compact set and $V \in C^1(\mathbb{R}^n; \mathbb{R})$ a function such that*

(1) $x - V'(x) \in K$, *for all* $x \in \partial K \cap \Psi$,
(2) $\langle F(x), V'(x)\rangle \geq 0$, *for all* $x \in K \cap \Psi$.

We set

$$E := \{x \in K \cap \Psi : \langle F(x), V'(x)\rangle = 0\}$$

and denote the largest invariant subset of E by $\mathcal{M}$. Then, for each $x_0 \in K$ such that $\gamma(x_0) \subset \Psi$, we have

$$\lim_{\tau \to +\infty} d(x(\tau; t_0, x_0), \mathcal{M}) = 0.$$

Proof. 1) Let us first remark that for x_0 given in K, the set $\Lambda(x_0)$ is invariant. Indeed, let $z \in \Lambda(x_0)$ be given. There exists $\{\tau_i\} \subset [t_0, +\infty)$ such that $\tau_i \to +\infty$ and $x(\tau_i; t_0, x_0) \to z$. Let $\tau \geq t_0$ be given. Using Theorem 2, we obtain $x(\tau; t_0, z) = \lim_{i\to\infty} x(\tau; t_0, x(\tau_i; t_0, x_0))$. Then, remarking from the uniqueness property of solutions that $x(\tau; t_0, x(\tau_i; t_0, x_0)) = x(\tau - t_0 + \tau_i; t_0, x_0)$, we get $x(\tau; t_0, z) = \lim_{i\to\infty} x(\tau - t_0 + \tau_i; t_0, x_0)$. Thus, setting $w_i := \tau - t_0 + \tau_i$, we see that $w_i \geq t_0, w_i \to +\infty$ and $x(w_i; t_0, x_0) \to x(\tau; t_0, z)$. It results that $x(\tau; t_0, z) \in \Lambda(x_0)$.

2) Let $x_0 \in K$ such that $\gamma(x_0) \subset \Psi$. We claim that there exists a constant $k \in \mathbb{R}$ such that

$$V(x) = k, \quad \forall x \in \Lambda(x_0).$$

Indeed, let $T > 0$ be given. We define the mapping $V^* : [t_0; +\infty) \to \mathbb{R}$ by the formula

$$V^*(t) := V(x(t; t_0, x_0));$$

The function $x(.) \equiv x(.; t_0, x_0)$ is absolutely continuous on $[t_0, t_0 + T]$, and thus V^* is a.e. strongly differentiable on $[t_0, t_0 + T]$. We have

$$\frac{dV^*}{dt}(t) = \langle V'(x(t)), \frac{dx}{dt}(t)\rangle, \quad \text{a.e. } t \in [t_0, t_0 + T].$$

We know by assumption that

$$x(t) \in K \cap \Psi, \forall t \geq t_0,$$

and

$$\langle \frac{dx}{dt}(t) + F(x(t)), v - x(t) \rangle \geq 0, \quad \forall v \in K, \ \text{ a.e. } t \geq t_0. \tag{17}$$

We claim that

$$\langle \frac{dx}{dt}(t), V'(x(t)) \rangle \leq 0, \quad \text{a.e. } t \geq t_0.$$

Indeed, if $x(t) \in \partial K$ then by assumption, $x(t) - V'(x(t)) \in K$ and setting $v = x(t) - V'(x(t))$ in (17), we obtain

$$\langle \frac{dx}{dt}(t), V'(x(t)) \rangle \leq -\langle F(x(t)), V'(x(t)) \rangle \leq 0, \quad \text{a.e. } t \geq t_0. \tag{18}$$

If $x(t) \in \ \text{int}\{K\}$, then there exists $\varepsilon > 0$ such that $x(t) - \varepsilon V'(x(t)) \in K$. Setting $v = x(t) - \varepsilon V'(x(t))$ in (17), we obtain

$$\varepsilon \langle \frac{dx}{dt}(t), V'(x(t)) \rangle \leq -\varepsilon \langle F(x(t)), V'(x(t)) \rangle \leq 0, \quad \text{a.e. } t \geq t_0 \tag{19}$$

and thus

$$\langle \frac{dx}{dt}(t), V'(x(t)) \rangle \leq 0, \quad \text{a.e. } t \geq t_0. \tag{20}$$

Thus,

$$\frac{dV^*}{dt}(t) \leq 0, \ \text{ a.e. } t \in [t_0, t_0 + T].$$

We know that $x \in C^0([t_0, t_0 + T]; \mathbb{R}^n)$, $\frac{dx}{dt} \in L^\infty(t_0, t_0 + T; \mathbb{R}^n)$ and $V \in C^1(\mathbb{R}^n; \mathbb{R})$. It follows that $V^* \in W^{1,1}(t_0, t_0 + T; \mathbb{R}^n)$ and applying Lemma 3.1 in [15], we obtain that V^* is decreasing on $[t_0, t_0 + T]$. The real T has been chosen arbitrarily and thus V^* is decreasing on $[t_0, +\infty)$. Moreover, Ψ is compact and thus V^* is bounded from below on $[t_0, +\infty)$. It results that

$$\lim_{\tau \to +\infty} V(x(\tau; t_0, x_0)) = k,$$

for some $k \in \mathbb{R}$.

Let $y \in \Lambda(x_0)$ be given. There exists $\{\tau_i\} \subset [t_0, +\infty)$ such that $\tau_i \to +\infty$ and $x(\tau_i; t_0, x_0) \to y$. By continuity,

$$\lim_{i \to +\infty} V(x(\tau_i; t_0, x_0)) = V(y).$$

Therefore, $V(y) = k$. Here, y has been chosen arbitrarily in $\Lambda(x_0)$ and thus

$$V(y) = k, \quad \forall y \in \Lambda(x_0).$$

3) The set $\gamma(x_0)$ is bounded and thus $\Lambda(x_0)$ is nonempty and

$$\lim_{\tau \to +\infty} d(x(\tau; t_0, x_0), \Lambda(x_0)) = 0.$$

Let us now check that $\Lambda(x_0) \subset E$. We first note that

$$\Lambda(x_0) \subset \overline{\gamma(x_0)} \subset \overline{K \cap \Psi} = K \cap \Psi.$$

We know from part (2) of this proof that there exists $k \in \mathbb{R}$ such that $V(x) = k, \forall x \in \Lambda(x_0)$. Let $z \in \Lambda(x_0)$ be given. Using Part (1) of this proof, we see that $x(t; t_0, z) \in \Lambda(x_0), \forall t \geq t_0$ and thus

$$V(x(t; t_0, z)) = k, \quad \forall t \geq t_0.$$

It results that

$$\frac{d}{dt} V(x(t; t_0, z)) = 0, \quad \text{a.e. } t \geq t_0. \tag{21}$$

Setting $x(.) \equiv x(.; t_0, z)$, we check as above that

$$\langle V'(x(t)), \frac{dx}{dt}(t) \rangle \leq -\langle F(x(t)), V'(x(t)) \rangle, \quad \text{a.e. } t \geq t_0. \tag{22}$$

Assumption (2) together with (21) and (22) yields

$$\langle F(x(t)), V'(x(t)) \rangle = 0, \quad \text{a.e. } t \geq t_0.$$

The mapping $t \to \langle F(x(t; t_0, z)), V'(x(t; t_0, z)) \rangle$ is continuous and thus taking the limit as $t \to t_0$, we obtain $\langle F(z), V'(z) \rangle = 0$. It results that $z \in E$.

Finally, $\Lambda(x_0) \subset \mathcal{M}$ since $\Lambda(x_0) \subset E$ and $\Lambda(x_0)$ is invariant. ■

Corollary 1. *Suppose that the assumption 1 holds. Let $V \in C^1(\mathbb{R}^n; \mathbb{R})$ be a function such that*

(1) $x - V'(x) \in K$, *for all* $x \in \partial K$.
(2) $\langle F(x), V'(x) \rangle \geq 0$, *for all* $x \in K$.
(3) $V(x) \to +\infty$ as $\|x\| \to +\infty, x \in K$.

We set

$$E := \{x \in K : \langle F(x), V'(x) \rangle = 0\},$$

and let $\mathcal{M}$ be the largest invariant subset of E. Then, for each $x_0 \in K$, the orbit $\gamma(x_0)$ is bounded and

$$\lim_{\tau \to +\infty} d(x(\tau; t_0, x_0), \mathcal{M}) = 0.$$

Proof. Let $x_0 \in K$ be given. We set

$$\Psi := \{x \in K : V(x) \leq V(x_0)\}.$$

The set Ψ is closed. Assumption (3) ensures that Ψ is bounded and thus Ψ is compact. If $\tau \geq t_0$, then $x(\tau; t_0, x_0) \in K$, and as in the proof of Theorem 4, we check that the mapping $t \to V(x(t; t_0, x_0))$ is decreasing on $[t_0, +\infty)$. Thus,

$$V(x(\tau; t_0, x_0)) \leq V(x(t_0; t_0, x_0)) = V(x_0)$$

and thus

$$\gamma(x_0) \subset \Psi.$$

It results that $\gamma(x_0)$ is bounded. From Theorem 4, we obtain

$$\lim_{\tau\to+\infty} d(x(\tau; t_0, x_0), \mathcal{M}') = 0,$$

where $\mathcal{M}'$ is the largest invariant subset of $\Psi \cap E$. It is clear that $\mathcal{M}' \subset \mathcal{M}$ and the conclusion follows. ∎

Example 1. Let us consider problem $P(t_0, x_0)$ with $F(.) = A.$, where

$$A = \begin{pmatrix} 0 & -1 \\ k & h \end{pmatrix}, \qquad k > 0, \qquad h > 0,$$

and $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$. We set

$$V(x_1, x_2) = \frac{1}{k}(\frac{1}{2}kx_1^2 + \frac{1}{2}x_2^2).$$

We check that both assumptions of Corollary 1 are satisfied. Indeed, $V'(x_1, x_2) = \frac{1}{k}(kx_1 \; x_2)^T$ and $\langle Ax, V'(x)\rangle = \frac{h}{k}x_2^2$. It is clear that assumptions 2 and 3 are satisfied. Moreover, $x - V'(x) = (0, \frac{k-1}{k}x_2) \in K$. Here $E = \{(x_1, 0); x_1 \geq 0\}$. Let $z = (z_1 \; 0)^T \in E$ be given. We claim that if $\gamma(z) \subset E$, then necessarily $z_1 = 0$. Indeed, suppose that $\gamma(z) \subset E$ and set $x(.) := x(., t_0, x_0)$. From the dynamics in E we have

$$-x_1(t)\dot{x}_1(t) \in N_{\mathbb{R}_+}(x_1(t)), \quad \text{a.e. } t \geq t_0$$

since $x_1 \geq 0$ and the right-hand-side is a cone, and

$$kx_1(t)v_2 \geq 0, \forall v_2 \in \mathbb{R}, \quad \text{a.e. } t \geq t_0.$$

From the first relation, we obtain $\frac{d}{dt}x_1^2(t) = 0, \quad$ a.e. $t \geq t_0$ from which it easily follows that $x_1(t) = z_1$ a.e. $t \geq t_0$. The second relation then gives $z_1 = 0$. Therefore, $\mathcal{M} = \{0\}$ is the largest invariant subset of E, and for any $x_0 \in K$, we have $\lim_{t\to+\infty} x(t; t_0, x_0) = 0$.

Example 2. Let us consider a system as in example 1; however, this time we choose the convex function $\Phi(x_1, x_2) = x_2^4$, so that $F(x) = Ax + \begin{pmatrix} 0 \\ 4x_2^3 \end{pmatrix}$. Using the same Lyapunov function, we get that $\langle F(x), V'(x_1, x_2)\rangle = \frac{h}{k}x_2^2 + \frac{4}{k}x_2^4$. Consequently, $E = \{(x_1, 0); x_1 \geq 0\}$. The same reasoning can be led as in example 1 to conclude that corollary 1 applies with $\mathcal{M} = \{0\}$.

Example 3. Let us finally consider $A = \begin{pmatrix} 0 & -1 \\ k & 0 \end{pmatrix}$, $k > 0$, $\Phi(x_1, x_2) = x_1^4$, so that $F(x) = Ax + \begin{pmatrix} 4x_1^3 \\ 0 \end{pmatrix}$. This time, still using the same Lyapunov function candidate, one gets $\langle F(x), V'(x_1, x_2)\rangle = x_1^4$. Therefore, $E = \{(0, x_2); x_2 \in \mathbb{R}\}$. From the dynamics in E, it follows that

$$-x_2(t)(v_1 - x_1(t)) \geq 0, \quad \forall\, v_1 \geq 0,$$

and

$$\dot{x}_2(t)(v_2 - x_2(t)) \geq 0, \quad \forall\, v_2 \in \mathbb{R}.$$

From the first relation, we obtain that $x_2(t) \in N_{\mathbb{R}^+}(0) = \mathbb{R}^-$, and from the second relation $\dot{x}_2(t) \in -N_{\mathbb{R}}(x_2) = \{0\}$. Let $z = (0, z_2)^T \in E$ be given. From the second relation, we obtain $x_2(t) = z_2$, a.e. $t \geq t_0$. From the first relation, we get $z_2 \leq 0$. Therefore, $\mathcal{M} = \{z |\ z_1 = 0, \text{and } z_2 \leq 0\}$ is the largest invariant subset of E and Corollary 1 applies.

Remark 4. The conclusion of Corollary 1 holds true if condition (3) is replaced by (3') K is bounded.

Corollary 2. *Suppose that assumptions 1 and 2 hold. Suppose that there exists* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1)
$$V(x) \geq a(\|x\|),\ x \in K,$$
with $a : \mathbb{R}_+ \to \mathbb{R}$ *satisfying* $a(0) = 0$, *a increasing on* $\mathbb{R}_+$;

(2) $V(0) = 0$;

(3) $x - V'(x) \in K$, *for all* $x \in \partial K$;

(4) $\langle F(x), V'(x)\rangle \geq 0$, *for all* $x \in K$;

(5) $E := \{x \in K : \langle F(x), V'(x)\rangle = 0\} = \{0\}$.

Then, the trivial solution of (7)–(8) is (a) the unique stationary solution of (7)–(8), (b) asymptotically stable, and (c) globally attractive, i.e., for each $x_0 \in K$, $\lim_{t\to+\infty} \|x(t; t_0, x_0)\| = 0$.

Proof. Assertion (a) is a consequence of Proposition 1. The stability is a direct consequence of Theorem 3. Moreover, we may apply Corollary 1 with $\mathcal{M} = \{0\}$ (since $E = \{0\}$) to obtain that for any $x_0 \in K$ the limit

$$\lim_{\tau\to+\infty} x(\tau; t_0, x_0) = 0$$

holds. Assertions (b) and (c) follow. ■

Corollary 3. *Suppose that the assumptions 1 and 2 hold. Suppose that there exists* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1)
$$V(x) \geq a(\|x\|),\ x \in K,$$
with $a : \mathbb{R}_+ \to \mathbb{R}$ *satisfying* $a(0) = 0$, *a increasing on* $\mathbb{R}_+$;

(2) $V(0) = 0$;

(3) $x - V'(x) \in K$, *for all* $x \in \partial K$;

(4) $\langle F(x), V'(x)\rangle \geq 0$, *for all* $x \in K$;

(5) $z \in K, z \neq 0 \Rightarrow \gamma(z) \cap E^c \neq \emptyset$,

where $E^c := \mathbb{R}^n \backslash E$ *and* $E := \{x \in K : \langle F(x), V'(x)\rangle = 0\}$. *Then, the trivial solution of (7)–(8) is (a) asymptotically stable, and (b) globally attractive.*

Proof. The stability is here also a direct consequence of Theorem 3. Moreover, we may also apply Corollary 1 with $\mathcal{M} = \{0\}$ to conclude. Indeed, assumption (5) entails that no solution can stay identically in E, other than the trivial solution. ■

Assumption (5) makes Corollary 3 quite similar to the original result by Krakovskii [33], [34]. The following result generalizes Theorem 3.

Corollary 4. *Suppose that assumptions 1 and 2 hold. Suppose that there exists* $\sigma > 0$ *and* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1)
$$V(x) \geq a(\|x\|),\ \ x \in K,\ \ \|x\| \leq \sigma,$$
with $a : [0, \sigma] \to \mathbb{R}$ *satisfying* $a(t) > 0$, $\forall t \in (0, \sigma)$;

(2) $V(0) = 0$;

(3) $x - V'(x) \in K$, *for all* $x \in \partial K$, $\|x\| \leq \sigma$;

(4) $\langle F(x), V'(x)\rangle \geq 0$, *for all* $x \in K$, $\|x\| \leq \sigma$;

(5) $E := \{x \in K, \|x\| \leq \sigma : \langle F(x), V'(x)\rangle = 0\} = \{0\}$.

Then, the trivial solution of (7)–(8) is (a) isolated in $\mathcal{S}$*, and (b) asymptotically stable.*

Proof. Assertion (a) is a direct consequence of Proposition 1. The stability follows from Theorem 3; the stability ensures the existence of $\delta > 0$ such that if $x_0 \in K$, $\|x\| \leq \delta$, then

$$\gamma(x_0) \subset B_\sigma := \{x \in \mathbb{R}^n : \|x\| \leq \sigma\}.$$

Applying Theorem 4 with $\Psi = B_\sigma$, we obtain for $x_0 \in K$, $\|x\| \leq \delta$ such that

$$\lim_{t\to+\infty} d(x(t; t_0, x_0), \mathcal{M}) = 0,$$

where $\mathcal{M}$ is the largest invariant subset of E. It is clear that assumption (5) yields $\mathcal{M} = \{0\}$. The attractivity and assertion (b) follow. ■

4. Linear Evolution Variational Inequalities

Let $K \subset \mathbb{R}^n$ be a closed convex set such that $0 \in K$. Let $A \in \mathbb{R}^{n\times n}$ be a given matrix. We consider Problem $P(t_0, x_0)$ with $F(\cdot) \equiv A.$, i.e., : Find $x \in C^0([t_0, \infty); \mathbb{R}^n)$ such that $\frac{dx}{dt} \in L^\infty_{\text{loc}}(t_0, +\infty; \mathbb{R}^n)$ and

$$\langle \frac{dx}{dt}(t) + Ax(t), v - x(t)\rangle \geq 0, \forall v \in K,\ \text{a.e. } t \geq t_0, \tag{23}$$
$$x(t) \in K, \quad t \geq t_0, \tag{24}$$
$$x(t_0) = x_0. \tag{25}$$

The $i\text{-}th$ canonical vector of $\mathbb{R}^n$ is denoted by $\bar{e}_i$. For a matrix $B \in \mathbb{R}^{n\times n}$, we set

$$E(K, B) := \{x \in K : \langle Bx, x\rangle = 0\} = \ker\{B + B^T\} \cap K.$$

Theorem 5. *Suppose that there exists a matrix $G \in \mathbb{R}^{n\times n}$ such that*

(1) $\inf_{x\in K\setminus\{0\}} \frac{\langle Gx,x\rangle}{\|x\|^2} > 0$,
(2) $\langle Ax, [G+G^T]x\rangle \geq 0, \forall x \in K$,
(3) $x \in \partial K \Rightarrow [I-[G+G^T]]x \in K$,
(4) $E(K,(G+G^T)A) = \{0\}$.

Then, the trivial solution of (23)–(24) is (a) the unique stationary solution of (23)–(24), (b) asymptotically stable, and (c) globally attractive.

Proof. Let $V \in C^1(\mathbb{R}^n;\mathbb{R})$ be defined by

$$V(x) = \frac{1}{2}\langle [G+G^T]x, x\rangle. \tag{26}$$

Then, $V'(x) = [G+G^T]x$, and we see that all the assumptions of Corollary 2 are satisfied. Indeed assumption (1) ensures the existence of a constant $k > 0$ such that $V(x) \geq k \parallel x \parallel^2, \forall x \in K$. It is clear that $V(0) = 0$. Finally, from assumptions (2), (3), and (4), we deduce that $\langle Ax, V'(x)\rangle \geq 0, \forall x \in K$, $x \in \partial K \Rightarrow x - V'(x) \in K$ and $E := \{x \in K : \langle Ax, V'(x)\rangle = 0\} = E(K,(G+G^T)A) = \{0\}$. ■

A matrix $A \in \mathbb{R}^{n\times n}$ satisfying the conditions (1), (2), and (3) in Theorem 5 is said to be *"Lyapunov positive semi-stable on K"*. This last concept has been introduced and studied in [9]. Using the results in [9], we remark that the conclusions of Theorem 5 hold for the following cases:

- A is positive definite.
- A is positive semi-definite and $E(K, A) = \{0\}$.
- A is strictly copositive on K (see [36, §2.3] for definition of copositive matrices).
- A is copositive on K and $E(K, A) = \{0\}$.
- K is a cone such that $x \in \partial K \Rightarrow x_i\bar{e}_i \in K$, and there exists a positive diagonal matrix D such that DA is copositive on K and $E(K, DA) = \{0\}$.
- A is Lyapunov positive stable on K (see [9]).

5. Nonlinear Variational Inequalities

Let $K \subset \mathbb{R}^n$ be a closed convex set such that $0 \in K$. Let $F \in C^1(\mathbb{R}^n;\mathbb{R})$ be a nonlinear mapping. We consider Problem $P(t_0, x_0)$: Find $x \in C^0([t_0,\infty);\mathbb{R}^n)$ such that $\frac{dx}{dt} \in L^\infty_{\text{loc}}(t_0,+\infty;\mathbb{R}^n)$ and

$$\langle \frac{dx}{dt}(t) + F(x(t)), v - x(t)\rangle \geq 0, \forall v \in K, \quad \text{a.e. } t \geq t_0, \tag{27}$$

$$x(t) \in K, \quad t \geq t_0, \tag{28}$$

$$x(t_0) = x_0. \tag{29}$$

Theorem 6. *Suppose that $F \in C^1(\mathbb{R}^n;\mathbb{R})$ is Lipschitz continuous and $F(0) = 0$. Let us here denote by $J_F(0)$ the Jacobian matrix of F at 0, i.e.,*

$$J_F(0) = \Big(\frac{\partial F_i}{\partial x_j}\Big).$$

Suppose that there exists a matrix $G \in \mathbb{R}^{n\times n}$ such that

(1) $\inf_{x\in K\setminus\{0\}} \frac{\langle Gx,x\rangle}{\|x\|^2} > 0$,
(2) $\inf_{x\in K\setminus\{0\}} \frac{\langle J_F(0)x,[G+G^T]x\rangle}{\|x\|^2} > 0$,
(3) $x \in \partial K \Rightarrow [I-[G+G^T]]x \in K$.

Then, the trivial solution of (27)–(28) is (a) isolated, and (b) asymptotically stable.

Proof. From Taylor's formula, we may write

$$F(x) = Ax + F_1(x),$$

where $A := J_F(0)$ and F_1 satisfies

$$\lim_{\|x\|\to 0} \frac{\| F_1(x) \|}{\| x \|} = 0 \tag{30}$$

Moreover, F_1 is Lipschitz continuous since $F_1(.) := F(.) - A.$, and F is assumed to be Lipschitz continuous.

Our aim is to verify that all conditions of Corollary 4 are satisfied with $V \in C^1(\mathbb{R}^n;\mathbb{R})$ defined by

$$V(x) = \frac{1}{2}\langle [G+G^T]x, x\rangle.$$

From assumption (1), we see that there exists a constant $c_1 > 0$ such that

$$V(x) = \langle Gx, x\rangle \geq c_1 \| x \|^2, \forall x \in K.$$

This yields assumption (1) of Corollary 4. It is clear that $V(0) = 0$ so that assumption (2) of Corollary 4 is also satisfied. Here, $V'(x) = [G+G^T]x$ and assumption (3) yields assumption (3) of Corollary 4. Finally, from assumption (2) we obtain that

$$\langle Ax, V'(x)\rangle \geq c_2 \| x \|^2, \quad \forall x \in K,$$

for some constant $c_2 > 0$. On the other hand, because of (30) there exists a constant $\sigma > 0$ such that

$$\| x \| \leq \sigma \Rightarrow \| F_1(x) \| \leq \frac{c_2}{\| G+G^T \|} \| x \| .$$

Thus, if $\| x \| \leq \sigma$, then

$$\begin{aligned}\langle F_1(x), V'(x)\rangle &= \frac{1}{2}\langle F_1(x), [G+G^T]x\rangle \\ &\geq -\frac{1}{2} \| G+G^T \| \| F_1(x) \| \| x \| \geq -\frac{1}{2}c_2 \| x \|^2 .\end{aligned}$$

It results that

$$\langle Ax + F_1(x), V'(x)\rangle \geq \frac{c_2}{2} \| x \|^2, \quad \forall x \in K, \| x \| \leq \sigma,$$

and thus assumptions (4) and (5) of Corollary 4 hold. ∎

The matrix $J_F(0) \in \mathbb{R}^{n\times n}$ satisfying the conditions (1), (2), and (3) in Theorem 6 is is said to be *"Lyapunov positive stable on K"*. This concept has also been introduced and studied in [9].

6. Applications

6.1. Absolute Stability

Let us consider the following system (see Fig. 2):

$$\begin{cases} \frac{dx}{dt}(t) = Ax(t) - By_L(t), \ \text{a.e. } t \geq 0, \\ y(t) = Cx(t), \\ y(t) \in K, \\ y_L(t) \in \partial\psi_K(y(t)), \end{cases} \tag{31}$$

where $K \subseteq \mathbb{R}^n$ is convex, closed, and $0 \in K$. Here, $\partial.$ denotes the subdifferential from convex analysis and $\psi_K(\cdot)$ is the indicator function of K [23]. We suppose also that a point $y_0 = CR^{-1}x_0 \in \text{int } K$ exists. Assume that (A, B, C) is a positive-real transfer matrix [17]. There exists $G = G^T$ positive definite and Q positive semi-definite such that $A^TG + GA = -Q$ and $GB = C^T$. Then, the dynamics in (31) can be rewritten equivalently as [9], [10]

$$\begin{cases} \langle \frac{dz}{dt}(t) - RAR^{-1}z(t), v - z(t)\rangle \geq 0, \forall v \in \bar{K}, \text{a.e.} \quad t \geq 0, \\ z(t) \in \bar{K}, \quad t \geq 0, \end{cases} \tag{32}$$

where $z = Rx$, R is a symmetric positive definite square root of G, and $\bar{K} = \{h \in \mathbb{R}^n : CR^{-1}h \in K\}$. One has

$$\langle GAx, x\rangle + \langle A^TGx, x\rangle = -\langle Qx, x\rangle, \forall x \in \mathbb{R}^n. \tag{33}$$

Thus,

$$\langle Ax, Gx\rangle = -\frac{1}{2}\langle Qx, x\rangle, \forall x \in \mathbb{R}^n. \tag{34}$$

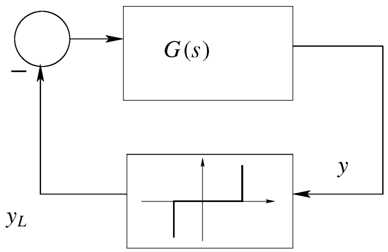

Fig. 2. PR transfer with multivalued feedback.

Consequently,

$$-\langle RAx, Rx\rangle \geq 0, \forall x \in \mathbb{R}^n. \tag{35}$$

Setting $z = Rx$, one gets

$$-\langle RAR^{-1}z, z\rangle \geq 0, \forall z \in \mathbb{R}^n. \tag{36}$$

Therefore, $-RAR^{-1}$ is positive semi-definite, and from the result in Section 4 we deduce

Corollary 5. *Let* (A, B, C) *be positive real. Let* $\mathcal{M}$ *be the largest invariant subset of* $ker\{RAR^{-1} + R^{-1}A^T R\} \cap \bar{K}$*; then, for each* $z_0 \in \bar{K}$*, we have* $\lim_{t\to+\infty} d(z(t; 0, z_0), \mathcal{M}) = 0$.

Corollary 6. *Let* (A, B, C) *be positive real. If* $ker\{RAR^{-1} + R^{-1}A^T R\} \cap \bar{K} = \{0\}$*, then the trivial solution of the system in (32) is asymptotically stable.*

Examples of systems in (31) are the so-called dissipative linear complementarity systems [18], [19], [20] with relative degree one, i.e.

$$\begin{cases} \frac{dx}{dt}(t) = Ax(t) + B\lambda, \\ 0 \leq y = Cx(t) \perp \lambda \geq 0, \end{cases} \tag{37}$$

where (A, B, C) is positive real. By elementary convex analysis, one can rewrite (37) as

$$-\frac{dx}{dt}(t) + Ax(t) \in B\partial\psi_{\mathbb{R}^m_+}(Cx(t)) \tag{38}$$

which is (31) with $K = \mathbb{R}^m_+$. Then, the above change of coordinates allows one to write (38) as the LEVI($-RAR^{-1}$, $\bar{K}$) similarly as in (32), with $\bar{K} = \{h \in \mathbb{R}^n : CR^{-1}h \geq 0\}$. Dissipative linear complementarity systems are models for electrical circuits with ideal diodes. As pointed out in the introduction, other applications exist [35].

6.2. *An Electrical Circuit*

Let us consider the circuit in Fig. 3 ($R_1, R_2, R_3 \geq 0$, $L_2, L_3 > 0$). One has $0 \leq -u_{D_4} \perp x_2 \geq 0$ and $0 \leq -u_{D_1} \perp -x_3 + x_2 \geq 0$, where u_{D_4} and u_{D_1} are the voltages of the diodes. The dynamical equations are the following ones

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\left(\frac{R_1+R_3}{L_3}\right)x_2 + \frac{R_1}{L_3}x_3 - \frac{1}{L_3C_4}x_1 + \frac{1}{L_3}\lambda_1 + \frac{1}{L_3}\lambda_2 \\ \dot{x}_3 = -\left(\frac{R_1+R_2}{L_2}\right)x_3 + \frac{R_1}{L_2}x_2 - \frac{1}{L_2}\lambda_1 \\ 0 \leq \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \perp \begin{pmatrix} -x_3 + x_2 \\ x_2 \end{pmatrix} \geq 0 \end{cases} \tag{39}$$

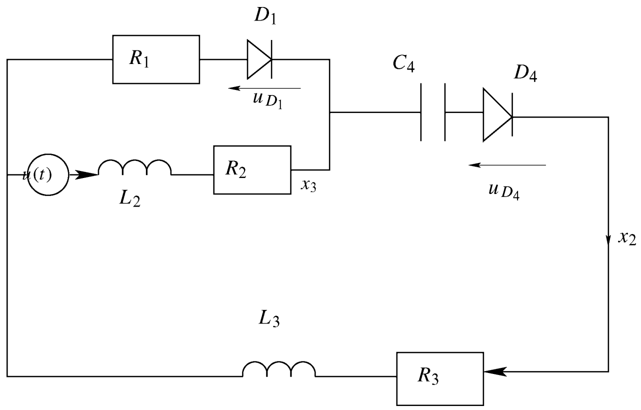

Fig. 3. A circuit with ideal diodes.

where $x_1(\cdot)$ is the time integral of the current across the capacitor, $x_2(\cdot)$ is the current across the capacitor, and $x_3(\cdot)$ is the current across the inductor L_2 and resistor R_2, $-\lambda_1$ is the voltage of the diode D_1 and $-\lambda_2$ is the voltage of the diode D_4. The system in (39) can be written compactly as $\dot{x} = Ax + B\lambda$, $0 \leq \lambda \perp y = Cx \geq 0$, with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{L_3 C_4} & -\frac{R_1+R_3}{L_3} & \frac{R_1}{L_3} \\ 0 & \frac{R_1}{L_2} & -\frac{R_1+R_2}{L_2} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ \frac{1}{L_3} & \frac{1}{L_3} \\ -\frac{1}{L_2} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We set

$$G = \begin{pmatrix} \frac{1}{C_4} & 0 & 0 \\ 0 & L_3 & 0 \\ 0 & 0 & L_2 \end{pmatrix}.$$

It is clear that G is symmetric and positive definite. Moreover, we see that $A^T G + GA = -Q$ with

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2(R_1 + R_3) & -2R_1 \\ 0 & -2R_1 & 2(R_1 + R_2) \end{pmatrix}.$$

The matrix Q is symmetric and positive semi-definite. Moreover, $GB = C^T$ and the system in (39) is positive real, as expected from the physics. From the previous subsection, we deduce that (39) can be rewritten as an evolution variational inequality as in (32), where R and z are easily calculated ($z_1 = \frac{1}{\sqrt{C_4}} x_1$, $z_2 = \sqrt{L_3} x_2$, $z_3 = \sqrt{L_2} x_3$). We have $\ker\{RAR^{-1} + R^{-1}A^T R\} = \{z \in \mathbb{R}^3 : z_2 = 0\}$ and $\bar{K} = \{z \in \mathbb{R}^3 :\ z_2 \geq 0,\ z_3 \leq \sqrt{\frac{L_2}{L_3}} z_2\}$. Thus,

$$E := \ker\{RAR^{-1} + R^{-1}A^T R\} \cap \bar{K}$$
$$= \{z \in \mathbb{R}^3 :\ z_2 = 0, z_3 \le 0\}.$$

Remark 5. The stationary points x^* have to satisfy

$$\begin{cases} 0 = -Ax^* + B\lambda^* \\ 0 \le \lambda^* \perp Cx^* \ge 0 \end{cases} \Leftrightarrow \begin{cases} -GAx^* + GB\lambda^* = 0 \\ 0 \le \lambda^* \perp Cx^* \ge 0 \end{cases} \Rightarrow \begin{cases} -x^{*T}GAx^* = 0 \\ 0 \le \lambda^* \perp Cx^* \ge 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} x^{*T}Qx^* = 0 \\ 0 \le \lambda^* \perp Cx^* \ge 0 \end{cases} \tag{40}$$

for some $\lambda^* \in \mathbb{R}^2$.

For example, let us now consider $R_1 = R_2 = 0, R_3 > 0$. Then, the set S of stationary points is given by

$$S = \{z \in \mathbb{R}^3 : z_1 \ge 0, z_2 = 0, z_3 \le 0\}.$$

The set S is an invariant subset of E. We claim that it is the largest one. Indeed, let us study the dynamics in E:

$$\begin{cases} \dot{z}_1 = 0, \\ 0 = -z_1 + \sqrt{C_4}\lambda_1 + \sqrt{C_4}\lambda_2, \\ \dot{z}_3 = -\frac{1}{\sqrt{L_2}}\lambda_1, \\ 0 \le \lambda \perp \begin{pmatrix} -z_3 \\ 0 \end{pmatrix} \ge 0. \end{cases} \tag{41}$$

Thus, in E we get $z_1(.) = z_1^*(= z_1(0))$, and from the second line in (41) it follows that $z_1^* \ge 0$. From the fourth line, we obtain $\lambda_1(t)z_3(t) = 0$, a.e. $t \ge 0$. Then, using the third line, we see that $\dot{z}_3 z_3(t) = 0$, a.e. $t \ge 0$. Thus, $\frac{1}{2}\frac{d}{dt} \mid z_3(t) \mid^2 = 0$, a.e. $t \ge 0$. It results that $z_3(.) = z_3^*(= z_3(0)) \le 0$. Thus, any invariant subset of E is a subset of S. From Corollary 5, for any $z_0 \in \bar{K}$, we have $\lim_{t \to +\infty} d(z(t; 0, z_0), S) = 0$.

Let us assume now that the diode D_4 is mounted in the opposite sense on Fig. 3. Then, $0 \le u_{D_4} \perp -x_2 \ge 0$, and the second line in (41) is changed to $0 = -z_1 + \sqrt{C_4}\lambda_1 - \sqrt{C_4}\lambda_2$. The only discrepancy with the previous case is that $z_1^* \le 0$.

7. Conclusions

This paper deals with the extension of the Krakovskii–LaSalle invariance lemma, to a class of nonsmooth dynamical systems. Evolution variational inequalities are the formalism of the work. Firstly some technical results (existence, uniqueness, continuous dependance in the initial conditions, invariance of the limit sets, etc.) are proven. Then, the invariance theorem and various corollaries are proposed. These results are applied to linear and nonlinear evolution variational inequalities. Several applications are proposed to illustrate the developments.

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