Absolute stability and the Lagrange–Dirichlet theorem with monotone multivalued mappings☆

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Abstract

This note presents an extension of the absolute stability problem and of the Lagrange–Dirichlet theorem, when the nonlinearities entering the model are considered within the class of monotone multivalued mappings (consequently including operators with piecewise-linear graphs that may represent physical effects like Coulomb friction, dead-zones, saturations, elasto-plasticity, and unilateral constraints).
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1. Introduction

The absolute stability problem (and its various forms known as the passivity theorems) and the Lagrange–Dirichlet theorem are both fundamental tools for the study of the stability of nonlinear dynamical systems [18,20,30]. The first one concerns the study of the stability of a linear positive real transfer function $G(s)$ (or more generally of any dissipative system) with a nonlinear static feedback characteristic (see Fig. 1). The second one gives a criterion for the stability of Lagrangian mechanical systems. In this note we first extend Theorem 5.6.18

in [30] [1] to the case when the static nonlinearities are multivalued monotone mappings (thus in particular allowing the nonlinearity to belong to the sector $[0, +\infty]$ and not restricted to the sector $[0, +\infty)$). Then we extend the Lagrange–Dirichlet theorem to the case of mechanical Lagrangian systems subject to unilateral [4] (or inequality) frictionless constraints. We make use of the convex analysis framework as developed by Rockafellar and Moreau [23,27]. Before stating the two main results, we briefly recall the conditions which guarantee that a given state x_0 is a fixed point of the dynamical system considered. They take the form of generalized equations. [2] The interest for extending the above stability tools towards the

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[1] $G(s)$ in Fig. 1 is a strictly positive real transfer matrix and the operator $PL : y \mapsto y_L$ satisfies $PL(0) = 0$, $y^T PL(y) \geqslant 0 \; \forall t \geqslant 0$ and $\forall y \in \mathbb{R}^m$, i.e. PL belongs to the sector $[0, +\infty)$.

[2] i.e. certain types of relations that are similar to equations except that one side of the relation is multivalued.
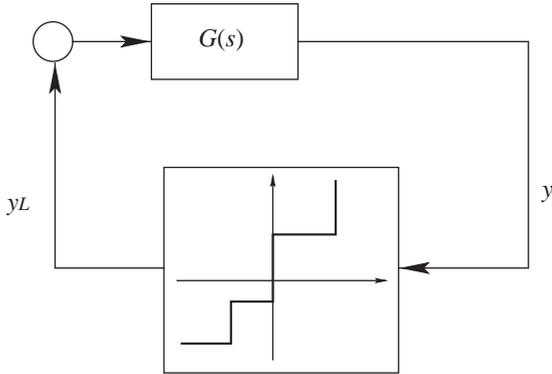
Fig. 1. Absolute stability with monotone multivalued mappings.

case when the feedback nonlinearity is a multivalued monotone mapping, lies in the fact that such operators are widely used in physical modelling (e.g. in mechanics [11–13,26]). Though monotonicity clearly implies dissipativity of the said operator, rendering such extensions more or less intuitively clear, the analysis has not, to the best of our knowledge, yet been done.

The note is organized as follows: in Section 2 we recall some basic concepts of convex analysis. Section 3 is devoted to the absolute stability problem, and Section 4 contains the material concerning the Lagrange–Dirichlet theorem applied to Lagrangian systems subject to unilateral constraints on the position. Conclusions end the note in Section 5.

2. Basic facts in convex and nonsmooth analysis

All the material in this section is taken from [23,27] and only aims at introducing the tools which are needed in this note.

Let $K \subset \mathbb{R}^n$ denote a convex set. Its indicator function is defined as

$$\psi_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{if } x \notin K. \end{cases} \tag{1}$$

A convex function $f(\cdot)$ satisfies $f(\lambda x+(1-\lambda)y) \leqslant \lambda f(x) + (1 - \lambda)f(y)$ for all $0 < \lambda < 1$, and for all x and y in its (convex) domain of definition. The subdifferential of a convex function $f(\cdot)$ at y is denoted as $\partial f(y)$ and is the set of subgradients, i.e. vectors γ satisfying $f(x) - f(y) \geqslant \gamma^T(x - y)$

for all x. The subdifferential of the indicator of K (which is convex if K is convex) is given by

$$\partial \psi_K(x) = \begin{cases} \{0\} & \text{if } x \in \text{Int}(K), \\ N_K(x) & \text{if } x \in \partial K, \\ \emptyset & \text{if } x \notin K, \end{cases} \tag{2}$$

where ∂K is the boundary of K, and $N_K(x) = \{z \mid z^T(\zeta - x) \leqslant 0, \forall \zeta \in K\}$ is the outwards normal cone to K at x. Notice that $0 \in N_K(x)$ and that we have drawn the sets $x + N_K(x)$ rather than $N_K(x)$ in Fig. 2. The set in (2) is the subdifferential from convex analysis. However when K is not convex it is possible to extend (2) using a different definition of the subdifferential (see [8, Proposition 2.4.12]). This will be useful when we deal with Lyapunov stability of mechanical systems with non-convex admissible configuration subspaces in Section 4.

Remark 1. The symbol ∂ is used in three different meanings in this paper: boundary of a set, subdifferential and partial derivative. Since these notations are classical we choose not to change them.

The inwards tangent cone $V(x)$ is the polar cone to $N_K(x)$ and is defined as $V(x) = \{z \mid \forall \zeta \in N_K(x), \zeta^T z \leqslant 0\}$. It is also sometimes denoted as $T_K(x)$ in the literature. Both the normal and the tangent cones are convex sets. If the set K is defined as $\{x \mid h(x) \geqslant 0\}$ for some differentiable function $h : \mathbb{R}^n \to \mathbb{R}^m$, then an alternative definition of the tangent cone at x is [24]

$$V(x) = \{v \in \mathbb{R}^n \mid v^T \nabla h_i(x) \geqslant 0, \ \forall \ i \in J(x)\} \tag{3}$$

with $J(x) = \{i \in \{1, \ldots, m\} \mid h_i(x) \leqslant 0\}$. One notes that this definition coincides with the first one as long as $x \in K$, and that K needs not be convex to define $V(x)$ in (3). We shall use some material from [24] in Section 4. Some examples are depicted in Fig. 2 (see also [4]).

A mapping $\rho(\cdot)$ from X to Y is said multivalued if it assigns to each element x of X a subset $\rho(x)$ of Y (which may be empty, contain just one element, or contain several elements). The mappings whose graphs are in Figs. 3(c)–(f) are multivalued [26]. A multivalued mapping $\rho(\cdot)$ is monotone if $(x - x')^T(y - y') \geqslant 0$ for any couples (x, y) and (x', y') in its graph. When $n = 1$ monotone mappings
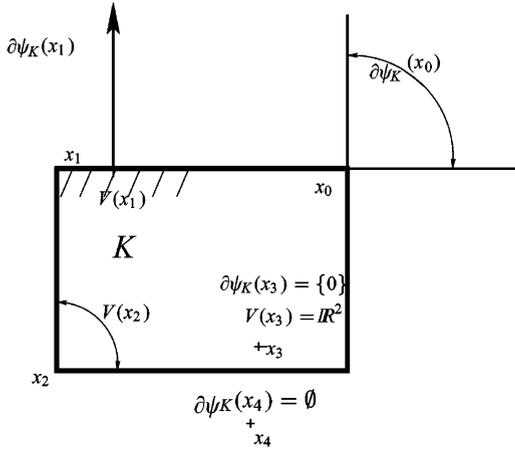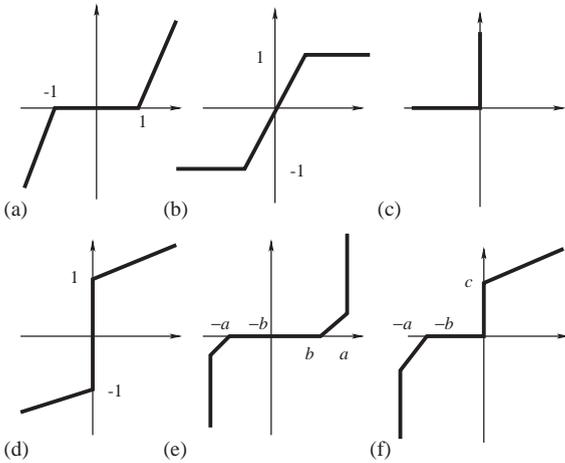
Fig. 2. Tangent and normal cones.

Fig. 3. Monotone mappings (one-dimensional case).

correspond to completely non-decreasing curves. Let $D(\rho) = \{x \mid x \in X, \rho(x) \neq \emptyset\}$ be the domain of $\rho(\cdot)$. A monotone mapping $\rho(\cdot)$ is maximal if for any $x \in X$ and any $y \in Y$ such that $\langle y - y_1, x - x_1 \rangle \geqslant 0$ for any $x_1 \in D(\rho)$ and any $y_1 \in \rho(x_1)$, then $y \in \rho(x)$. Complete nondecreasing curves in $\mathbb{R}^2$ are the graphs of maximal monotone mappings. Another interpretation is that the graph of a maximal monotone mapping cannot be enlarged without destroying the monotonicity (hence the notion of maximality). The subdifferential $\partial\varphi(\cdot)$ of a convex lower semicontinuous function on $\mathbb{R}^n$ is a monotone mapping. Examples of monotone

mappings ($n=1$) are depicted in Fig. 3. They may represent various physical laws, like dead-zone (a), saturation or elasto-plasticity (b), corner law—unilateral effects, ideal diode characteristic—(c), Coulomb friction (d), MOS transistor ideal characteristic (e), unilateral and adhesive effects (f) [11]. One has for instance $\varphi(x) = \psi_{\mathbb{R}^-}(x)$ in Fig. 3(c), $\varphi(x) = |x| + x^2/2$ for (d),

$$\varphi(x) = \psi_{(-\infty, a]}(x) - \psi_{[-a, +\infty)}(x)$$
$$+ \begin{cases} \frac{1}{2}(a - b)(x - b)^2 & \text{if } |x| \geqslant b \\ 0 & \text{if } |x| < b \end{cases} \quad \text{for (e).}$$

If $\varphi(x_1, \ldots, x_m) = \mu_1|x_1| + \cdots + \mu_m|x_m| + \frac{1}{2}x^\mathrm{T}x$, then $\partial\varphi(0) = ([-\mu_1, \mu_1], \ldots, [-\mu_m, \mu_m])^\mathrm{T}$.

3. The absolute stability problem

The absolute stability problem consists of studying the stability of a system as in Fig. 1 where $G(s)$ is a positive real transfer function and the feedback branch contains a sector static nonlinearity. It is of interest to extend this result to the case where the operator $PL : y \in \mathbb{R}^l \mapsto y_L \in \mathbb{R}^l$ is a monotone operator.

The state-space equations of the system in Fig. 1 are given by

$$\dot{x} \stackrel{\text{a.e.}}{=} Ax - By_L,$$

$$y = Cx,$$

$$y_L \in \partial\varphi(y), \tag{4}$$

where $y, y_L \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and a.e. means almost everywhere in the Lebesgue measure. The fixed points of (4) can be characterized with the generalized equation $0 \in \{Ax_0\} - B\partial\varphi(Cx_0)$. Let us assume that

(a) $G(s) = C(sI - A)^{-1}B$, with (A, B, C) a minimal representation, is a strictly positive real (SPR) transfer matrix. In particular this implies that there exists positive definite matrices $P = P^\mathrm{T}$ and $Q = Q^\mathrm{T}$ such that $PA + A^\mathrm{T}P = -Q$ and $B^\mathrm{T}P = C$ [18].
(b) $\varphi : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is convex lower semicontinuous, so that $\partial\varphi$ is a maximal monotone multivalued mapping (see e.g. [3, Example 2.3.4]).

Lemma 1. *Let assumptions* (a) *and* (b) *hold. If* $Cx(0) \in \text{dom}\, \partial\varphi$, *then the system in* (4) *has a unique absolutely continuous* (*AC*) *solution on* $[0, +\infty)$.

Proof. Let R be the square root of P, i.e. $R = R^{\mathrm{T}} > 0$, $RR = P$. Consider the convex lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by $f(z) = \varphi(CR^{-1}z)$. Using (a) shows that $\text{Ker}(C^{\mathrm{T}}) = \{0\}$ so that $\text{Im}(CR^{-1}) = \text{Im}(C) = \mathbb{R}^m$. From [27, Theorem 2.3.9] it follows that $\partial f(z) = R^{-1}C^{\mathrm{T}}\partial\varphi(CR^{-1}z)$. Let us prove that the system

$$\dot{z} \in RAR^{-1}z - \partial f(z),$$

$$z(0) = Rx(0), \tag{5}$$

has a unique AC solution on $[0, +\infty)$. First, to say that $Cx(0) \in \text{dom}\, \partial\varphi$ is to say that $CR^{-1}z(0) \in \text{dom}\, \partial\varphi$, and this just means that $z(0) \in \text{dom}\, \partial f$. Second, it follows from the Kalman–Yakubovich–Popov lemma that $RAR^{-1} + (RAR^{-1})^{\mathrm{T}}$ is negative definite [18, Section 3.3.1]. Therefore the multivalued mapping $-RAR^{-1} + \partial f$ is maximal monotone [3, Lemma 2.4]. Consequently the existence and uniqueness result follows from [13, Theorem 3.1].

Now set $x(t) = R^{-1}z(t)$. It is straightforward to check that $x(t)$ is a solution of the system in (4). Actually the system in (5) is deduced from (4) by the change of state vector $z = Rx$. $\square$

As an example, let us consider dissipative linear complementarity systems (LCS) [7]:

$$\dot{x} = Ax + B\lambda,$$

$$0 \leqslant y = Cx \perp \lambda \geqslant 0, \tag{6}$$

where (A, B, C) satisfies (a) above, $y, \lambda \in \mathbb{R}^m$, and $Cx(0) \geqslant 0$. The second line in (6) is a set of complementarity conditions between y and λ, stating that both these terms have to remain non-negative and orthogonal one to each other. The LCS in (6) can be equivalently rewritten as in (5) with $\varphi(y) = \psi_{(\mathbb{R}^+)^m}(y)$.

Remark 2. It is interesting to note a close connection between the system in (5) and Moreau's first order sweeping process [16,22], i.e. evolution problems of the form $\dot{x}(t) \in -\partial\psi_{C(t)}(x(t))$ for some set $C(t)$. The systems we will deal with in the next section are second order sweeping processes. Incidentally Lemma 1 sheds a new light on the result in [7] since it shows that

initial jumps in the solution $x(t)$ are not needed if $x(0)$ belongs to the admissible set $\{x \mid Cx \in \text{dom}\, \partial\varphi\}$. The results in [7] allow a feedthrough term in y and provide necessary and sufficient conditions for absence of initial jumps. The present paper does not allow a feedthrough term and gives only sufficient conditions for non initial jumps, however more general functions $\varphi(\cdot)$ are considered. Finally Lemma 1 is related to the uniqueness result in [17] where the graph $\partial\varphi$ is a relay characteristic and the transfer matrix $G(s)$ is a P-matrix [9] for sufficiently large $s \in \mathbb{R}$. SPR transfer matrices satisfy such requirements, but we considered more general characteristics.

Finally let us note in passing that Lemma 1 applies to nonlinear systems as $\dot{x} = -\sum_{k=0}^{n} x^{2k+1} - y_L$, $y = x$, $y_L \in \partial\varphi(y)$, $x \in \mathbb{R}$. Indeed the dynamics $-y_L \mapsto y$ is strictly dissipative with storage function $x^2/2$, so that $P = 1$ and $z = x$.

Let us notice that $y \in \text{dom}\, \partial\varphi$. Finally there exists a Lebesgue integrable function $w(t)$ such that $x(t) = \int w(\tau)\,\mathrm{d}\tau$, where $\mathrm{d}\tau$ is the Lebesgue measure. Hence $\mathrm{d}x = w(t)\,\mathrm{d}t$ as an equality of measures.

Lemma 2. *Let assumptions* (a), (b) *hold, the initial data be such that* $Cx(0) \in \text{dom}\, \partial\varphi$, *and assume that the graph of* $\partial\varphi$ *contains* $(0, 0)$. *Then*: (i) $x = 0$ *is the unique solution of the generalized equation* $Ax \in B\partial\varphi(Cx)$. (ii) *The fixed point* $x = 0$ *of the system in* (4) *is exponentially stable.*

Proof. The proof of part (i) is as follows. First of all notice that $x = 0$ is indeed a fixed point of the dynamics with no control, since $0 \in B\partial\varphi(0)$. Now $Ax \in B\partial\varphi(Cx) \Rightarrow PAx \in PB\partial\varphi(Cx) \Rightarrow x^{\mathrm{T}}PAx = x^{\mathrm{T}}\partial g(x)$, where $g(x) = \varphi(Cx)$ [27, p. 225], $g(\cdot)$ is convex [27, p. 38], and we used assumption (a). The multivalued mapping $\partial g(x)$ is monotone since $g(\cdot)$ is convex. Thus $x^{\mathrm{T}}\partial g(x) \geqslant 0$ for all $x \in \mathbb{R}^n$. Now there exists $Q = Q^{\mathrm{T}} > 0$ such that $x^{\mathrm{T}}PAx = -\frac{1}{2}x^{\mathrm{T}}Qx < 0$ for all $x \neq 0$. Clearly then x satisfies the generalized equation only if $x = 0$.

Let us now prove part (ii). Consider the candidate Lyapunov function $W(x) = \frac{1}{2}x^{\mathrm{T}}Px$. From Lemma 1 it follows that the dynamics in (4) possesses on $[0, +\infty)$ a solution $x(t)$ which is AC, and whose derivative $\dot{x}(t)$ exists a.e.. The same applies to W which is AC [28, p. 189]. Differentiating along the closed-loop trajectories

we get

$$\frac{\mathrm{d}W}{\mathrm{d}t}(x(t)) \overset{\mathrm{a.e.}}{=} x^{\mathrm{T}}Pw$$

$$= x^{\mathrm{T}}P(Ax - By_L) = -x^{\mathrm{T}}Qx - x^{\mathrm{T}}PBy_L$$

$$= -x^{\mathrm{T}}Qx - x^{\mathrm{T}}C^{\mathrm{T}}y_L, \tag{7}$$

where y_L is any vector that belongs to $\partial\varphi(Cx)$. The equality in the first line means that the density of the measure $\mathrm{d}(W \circ x)$ with respect to the Lebesgue measure $\mathrm{d}t$ (which exists since $W(x(t))$ is AC) is the function $x^{\mathrm{T}}Pw$. Consequently $\mathrm{d}W/\mathrm{d}t + x^{\mathrm{T}}Qx \in -x^{\mathrm{T}}C^{\mathrm{T}}\partial\varphi(Cx) = -x^{\mathrm{T}}\partial g(x)$ a.e.. Let us consider any $z \in \partial g(x)$. One gets $\mathrm{d}W/\mathrm{d}t \overset{\mathrm{a.e.}}{=} -x^{\mathrm{T}}Qx - x^{\mathrm{T}}z \leqslant -x^{\mathrm{T}}Qx$ from the property of monotone multivalued mappings and since $(x, z) = (0, 0)$ belongs to the graph of $\partial g(x)$. The set of time instants at which the inequality $\mathrm{d}W/\mathrm{d}t \leqslant -x^{\mathrm{T}}Qx$ is not satisfied is negligible in the Lebesgue measure. It follows that the function of time $W(\cdot)$, which is continuous, is nonincreasing. Thus one has $W(t) - W(0) = \int_0^t (-x^{\mathrm{T}}Qx - x^{\mathrm{T}}z)\,\mathrm{d}\tau \leqslant -\int_0^t x^{\mathrm{T}}Qx\,\mathrm{d}\tau$. Consequently $\frac{1}{2}\lambda_{\min}(P)x^{\mathrm{T}}x \leqslant W(0) - \int_0^t \lambda_{\min}(Q)x^{\mathrm{T}}x\,\mathrm{d}\tau$, where $\lambda_{\min}(\cdot)$ is the smallest eigenvalue. By the Gronwall's lemma one gets that $\frac{1}{2}\lambda_{\min}(P)x^{\mathrm{T}}x \leqslant W(0)\exp(-2(\lambda_{\min}(Q)/\lambda_{\min}(P))t)$ which concludes the proof. $\quad\square$

Remark 3. Related works can be found in [14] and in [21, Theorem 10.5.31], which appears as a particular case of Lemma 2. Other extensions for the passivity theorem with feedback branches that may contain various operators and nonlinearities, can be found in [20]. Some of the operators $y \mapsto y_L$ considered in [20] encompass the ones considered in this paper (like the saturation in Fig. 3(b)). However multivalued mappings are not included in the framework of [20], and neither are graphs including unilateral constraints as in Figs. 3(c), (e), and (f).

4. The Lagrange–Dirichlet theorem

Let us consider the following class of unilaterally constrained mechanical systems:

$$M(q)\ddot{q} + F(q, \dot{q}) = \nabla h(q)\lambda,$$

$$q(0) = q_0, \dot{q}(0^-) = \dot{q}_0,$$

$$0 \leqslant h(q) \perp \lambda \geqslant 0,$$

$$\dot{q}(t_k^+) = -e\dot{q}(t_k^-) + (1+e)\mathrm{prox}_{M(q(t_k))}$$

$$\times [\dot{q}(t_k^-), V(q(t_k))]. \tag{8}$$

In (8) $M(q) = M^{\mathrm{T}}(q) > 0$ is the $n \times n$ inertia matrix, $F(q, \dot{q}) = C(q, \dot{q})\dot{q} + (\partial U/\partial q)(q)$, where $C(q, \dot{q})\dot{q}$ denotes centripetal and Coriolis generalized forces, whereas $U(q)$ is a smooth potential energy from which conservative forces derive, and $h(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^m$. We assume that $h(q_0) \geqslant 0$. The impact times are generically denoted as t_k, the left-limit $\dot{q}(t_k^-) \in -V(q(t_k))$ whereas the right-limit $\dot{q}(t_k^+) \in V(q(t_k))$. The third line in (8) is a collision mapping that relates pre- and post-impact generalized velocities, and $e \in [0, 1]$ is a restitution coefficient [19]. The notation $\mathrm{prox}_{M(q)}$ means the proximation in the kinetic metric, i.e. the metric defined as $x^{\mathrm{T}}M(q)y$ for $x, y \in \mathbb{R}^n$: the vector $(\dot{q}(t_k^+) + e\dot{q}(t_k^-))/(1+e)$ is the closest vector to the pre-impact velocity, inside $V(q(t_k))$ (it can therefore be computed through a quadratic programme) [24]. In particular the impact law in (8) implies that the kinetic energy loss at time t_k satisfies (see [2; 4, p. 199, 489; 19])

$$T_L(t_k) = -\frac{1}{2}\frac{1-e}{1+e}(\dot{q}(t_k^+)$$

$$-\dot{q}(t_k^-))^{\mathrm{T}}M(q(t_k))(\dot{q}(t_k^+) - \dot{q}(t_k^-)) \leqslant 0. \tag{9}$$

Remark 4. The formulation of the unilateral constraints in (8) does not encompass all closed domains $\Phi = \{q \mid h(q) \geqslant 0\}$, as simple non-convex cases with so-called reentrant corners prove [5]. It can be used to describe admissible domains Φ which are defined either by a single constraint (i.e. $m = 1$), or with $m < +\infty$ where convexity holds at nondifferentiable points of the boundary $\partial\Phi$ (such sets are called *regular* [8]). It is easy to imagine physical examples that do not fit within this framework, e.g. a ladder. We prefer not to analyze this case in this note since neither the modelling nor the mathematical parts are yet fixed for non-differentiable non-convex boundaries $\partial\Phi$ [12, p. 156; 13, Section 9.5].

Let us note that the tangent cone $V(q(t))$ is assumed to have its origin at $q(t)$ so that $0 \in V(q(t))$ to allow for post-impact velocities tangential to the admissible set boundary $\partial \Phi$. The second line in (8) is a set of complementarity conditions between $h(q)$ and λ, stating that both these terms have to remain non-negative and orthogonal one to each other. Let us make the following assumptions:

Assumption 1. The gradients $\nabla h_i(q) = \partial h / \partial q^{\mathrm{T}}(q)$ are not zero at the contact configurations $h_i(q) = 0$, and the vectors ∇h_i, $1 \leqslant i \leqslant m$, are independent. Furthermore the functions $h(\cdot)$, $F(q, \dot{q})$, $M(q)$ and the system's configuration manifold are real analytic, and $\|F(q, \dot{q})\|_q \leqslant d(q, q(0)) + \|\dot{q}\|_q$, where $d(\cdot, \cdot)$ is the Riemannian distance and $\|\cdot\|_q$ is the norm induced by the kinetic metric.

Then the following results hold, which are essentially a compilation of Proposition 32, Theorems 8 and 10, and Corollary 9 of [2]:

(i) Solutions of (8) exist on $[0, +\infty)$ such that $q(\cdot)$ is AC, whereas $\dot{q}(\cdot)$ is right-continuous of local bounded variation (RCLBV). In particular the left- and right-limits of these functions exist everywhere.

(ii) The function $q(\cdot)$ cannot be supposed to be everywhere differentiable. One has $q(t) = q(0) + \int_0^t v(s) \, \mathrm{d}s$ for some function $v(\cdot) \overset{\text{a.e.}}{=} \dot{q}(\cdot)$. Moreover $\dot{q}(t^+) = v(t^+)$ and $\dot{q}(t^-) = v(t^-)$ [16].

(iii) Solutions are unique (however, in general, they do not depend continuously on the initial conditions).

(iv) The acceleration $\ddot{q}$ is a *measure* $\mathrm{d}v$, which is the sum of two measures: an atomic measure $\mathrm{d}\mu_a$, and a Lebesgue integrable function which we denote $\ddot{q}(\cdot)$, i.e. $\mathrm{d}v = \mathrm{d}\mu_a + \ddot{q}(t) \, \mathrm{d}t$. The atoms correspond to the impact times [24]. See Remark 5 for some comments on this decomposition.

(v) The set of impact times is countable. In many applications one has $\mathrm{d}\mu_a = \sum_{k \geqslant 0} [\dot{q}(t_k^+) - \dot{q}(t_k^-)] \delta_{t_k}$, where δ_t is the Dirac measure and the sequence $\{t_k\}_{k \geqslant 0}$ can be ordered, i.e. $t_{k+1} > t_k$. However phenomena like accumulations of left-accumulations of impacts may exist (at least bounded variation does not preclude them). In any case the ordering may not be possible. This

is a sort of complex Zeno behaviour.[3] In the case of elastic impacts ($e = 1$) it follows from [2, Proposition 4.11] that $t_{k+1} - t_k \geqslant \delta > 0$ for some $\delta > 0$. Hence, solutions are piecewise continuous in this case.

(vi) Any quadratic function $W(\cdot)$ of $\dot{q}$ is itself RCLBV, hence its derivative is a measure $\mathrm{d}W$ [24]. Consequently $\mathrm{d}W \leqslant 0$ has a meaning and implies that the function $W(\cdot)$ does not increase [10, p. 101].

These results enable one to lead a stability analysis safely. Let us now introduce a new formulation of the dynamics in (8), which can be written as the following measure differential inclusion (MDI) [24]

$$-M(q(t)) \, \mathrm{d}v - F(q(t), v(t^+)) \, \mathrm{d}t \in \partial \psi_{V(q(t))}(w(t))$$

$$\subseteq \partial \psi_{\Phi}(q(t)), \tag{10}$$

where $w(t) = (v(t^+) + ev(t^-))/(1 + e) \in \partial V(q(t))$ from (8). If $e = 0$ then $w(t) = v(t^+)$, if $e = 1$ then $w(t) = (v(t^+) + v(t^-))/2$. Moreover when $v(\cdot)$ is continuous then $w(t) = v(t)$. The term MDI has been coined by Moreau, and (10) may also be called Moreau's second-order sweeping process [16]. The inclusion in the right-hand side of (10) is proved in Appendix A. When $\dot{q}(t)$ is discontinuous, (10) implies that Moreau's collision rule in (8) is satisfied. This can be proved using basic tools from convex analysis, see Appendix B. The term $\psi_{V(q(t))}(w(t))$ can be interpreted as a velocity potential and its subdifferential $\partial \psi_{V(q(t))}(w(t))$ is depicted in Fig. 4 in special cases. The MDI in (10), whose left-hand side is a measure and whose right-hand side is a cone, has the following meaning [19,22]: there exists a positive measure $\mathrm{d}\mu$ such that both $\mathrm{d}t$ and $\mathrm{d}v$ possess densities with respect to $\mathrm{d}\mu$, denoted respectively as $(\mathrm{d}t/\mathrm{d}\mu)(\cdot)$ and $(\mathrm{d}v/\mathrm{d}\mu)(\cdot)$. One also has

$$\frac{\mathrm{d}t}{\mathrm{d}\mu}(t) = \lim_{\varepsilon \to 0, \, \varepsilon > 0} \frac{\mathrm{d}t([t, t + \varepsilon])}{\mathrm{d}\mu([t, t + \varepsilon])}$$

[22, p. 9; 25], which shows the link with the usual notion of a derivative. The choice of $\mathrm{d}\mu$ is not unique

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[3] I.e. all phenomena involving an infinity of events in a finite time interval, and which occur in various types of hybrid systems like Filippov's inclusions, etc. [29].
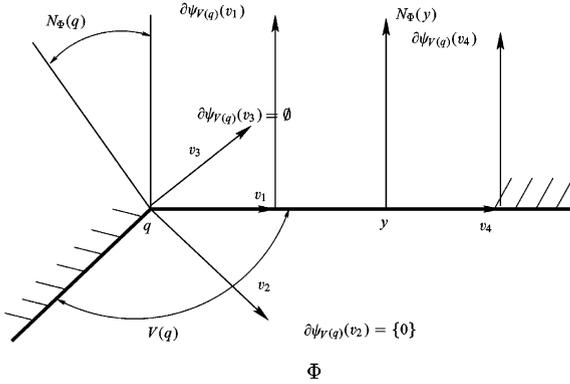
Fig. 4. Moreau's velocity potential.

because the right-hand side is a cone [24]. However by the Lebesgue–Radon–Nikodym theorem [28], the densities $(\mathrm{d}t/\mathrm{d}\mu)(\cdot)$ and $(\mathrm{d}v/\mathrm{d}\mu)(\cdot)$ are unique functions for a given $\mathrm{d}\mu$. To shed some light on this, let us consider for instance $\mathrm{d}\mu = \mathrm{d}t + \Sigma_{k \geqslant 0}\delta_{t_k}$, which corresponds to applications where the system is subject to impacts at times t_k and otherwise evolves freely. Then $(\mathrm{d}t/\mathrm{d}\mu)(t_k) = 0$ (the Lebesgue measure $\mathrm{d}t$ and the Dirac measure δ_t are mutually singular) whereas $(\mathrm{d}v/\mathrm{d}\mu)(t_k) = v(t_k^+) - v(t_k^-)$ (t_k is an atom of the measure $\mathrm{d}v$). When $t \neq t_k$ then $(\mathrm{d}t/\mathrm{d}\mu)(t) = 1$ and $(\mathrm{d}v/\mathrm{d}\mu)(t) = \dot{v}(t)$.

Therefore the meaning of (10) is that there exists a positive measure $\mathrm{d}\mu$ with respect to which both $\mathrm{d}t$ and $\mathrm{d}v$ possess densities, and

$$-M(q(t))\frac{\mathrm{d}v}{\mathrm{d}\mu}(t) - F(q(t), v(t^+))\frac{\mathrm{d}t}{\mathrm{d}\mu}(t)$$

$$\in \partial\psi_{V(q(t))}(w(t)) \subseteq \partial\psi_\Phi(q(t)) \qquad (11)$$

holds $\mathrm{d}\mu$—almost everywhere. In a sense, densities replace derivatives, for measures. When dealing with measure differential equations or inclusions, it is then natural to manipulate densities instead of derivatives. In general one can choose $\mathrm{d}\mu = |\mathrm{d}v| + \mathrm{d}t$ [22, p. 90], where $|\mathrm{d}v|$ is the absolute value of $\mathrm{d}v$, or $\mathrm{d}\mu = \|v(t)\|\mathrm{d}t + \mathrm{d}\mu_a$, or $\mathrm{d}\mu = \mathrm{d}t + \mathrm{d}\mu_a$. It is fundamental to recall at this stage, that the solution of (11) does not depend on this choice. For instance, if $\mathrm{d}\mu = \|v(t)\|\mathrm{d}t + \mathrm{d}\mu_a$ then for all $t \neq t_k$, $(\mathrm{d}t/\mathrm{d}\mu)(t) = 1/\|v(t)\|$ and $(\mathrm{d}v/\mathrm{d}\mu)(t) = \ddot{q}(t)/\|v(t)\|$. Whereas if $\mathrm{d}\mu = \mathrm{d}t + \mathrm{d}\mu_a$ then for all $t \neq t_k$, $(\mathrm{d}t/\mathrm{d}\mu)(t)=1$ and $(\mathrm{d}v/\mathrm{d}\mu)(t)=\ddot{q}(t)$.

Remark 5. The above mathematical framework is more than just a mathematical fuss. Indeed as noted in [24], introducing the velocity into the right-hand side of the dynamics as done in (10), not only allows one to get a compact formulation of the nonsmooth dynamics (see Fig. 6 in this respect), but it also paves the way towards the consideration of friction in the model. In turn it is clear that introducing friction, is likely to complicate the dynamics. Especially, the above framework paves the way towards more complex cases where the measure $\mathrm{d}v$ may contain a third term $\mathrm{d}\mu_{na}$ which is a nonatomic measure singular with respect to the Lebesgue measure $\mathrm{d}t$ (Assumption 1 implies that $\mathrm{d}\mu_{na} = 0$ [2]). In summary the dynamics in (11) is rich enough to encompass complex behaviours involving solutions which may be far from merely piecewise continuous as for the models and stability analysis in [1]. This is a consequence of replacing functions by the more general notion of measure, at the price of a more involved model. In fact using measures allows one to encompass somewhat complex Zeno behaviours occurring in unilaterally constrained mechanical systems in a rigorous manner.

In the case of unconstrained Lagrangian mechanical systems, the Lagrange–Dirichlet theorem states that the equilibrium point $(q^*, 0)$ is locally stable if the potential energy $U(q)$ has a strict minimum at q^*. First notice that since $F(q,0) = \partial U/\partial q$ and $0 \in V(q)$, fixed points of (10) satisfy the generalized equation $0 \in \partial\psi_\Phi(q^*)+(\partial U/\partial q)(q^*)$ which in particular implies $q^* \in \Phi$. Conditions under which such a generalized equation possess at least one solution, and numerical algorithms to compute one solution, exist [15]. In the following, we shall assume for simplicity that the solutions are isolated.

Lemma 3. *Consider a mechanical system as in* (8). *Then if* $\psi_\Phi(q)+U(q)$ *has a strict minimum at* q^*, *the equilibrium point* $(q^*, 0)$ *is Lyapunov stable.*

Let us note that Φ need not be convex in general (for instance, the equilibrium may exist in $\mathrm{Int}(\Phi)$, or it may belong to $\partial\Phi$ but be forced by the continuous dynamics, see Fig. 5 for planar examples with both convex and non-convex Φ; it is obvious that in the depicted non-convex case all points $(q^*, 0)$ with $q^* \in \partial\Phi$ are fixed points of the dynamics).
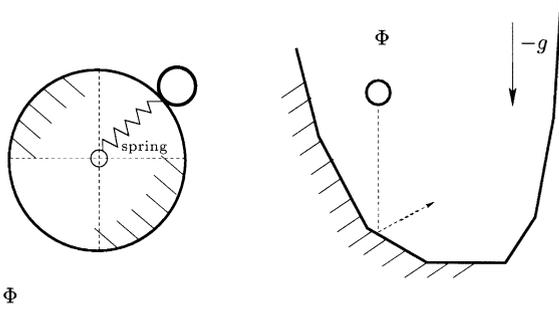
Fig. 5. Convex and nonconvex admissible sets.

Proof. The proof may be led as follows. Let us consider the nonsmooth Lyapunov candidate function

$$W(q, \dot{q}) = \tfrac{1}{2}\dot{q}^{\mathrm{T}}M(q)\dot{q} + \psi_\Phi(q) + U(q) - U(q^*).$$
(12)

Since the potential $\psi_\Phi(q) + U(q)$ has a strict minimum at q^* equal to $U(q^*)$, this W is positive definite on the whole state space. Also $W(q, \dot{q}) \leqslant \beta(\|q\|, \|\dot{q}\|)$ for some class K function $\beta(\cdot)$ is satisfied on Φ ($\ni q(t)$ for all $t \geqslant 0$). The potential function $\psi_\Phi(q) + U(q)$ is continuous on Φ. Thus $W(q, \dot{q})$ in (12) satisfies the requirements of a Lyapunov function candidate on Φ, despite the indicator function has a discontinuity on $\partial\Phi$ (but is continuous on the closed set Φ, see (1)). Moreover since (10) secures that $q(t) \in \Phi$ for all $t \geqslant 0$, it follows that $\psi_\Phi(q(t)) = 0$ for all $t \geqslant 0$. In view of this one can safely discard the indicator function in the subsequent stability analysis. Let us examine the variation of $W(q, \dot{q})$ along trajectories of (11). In view of the above discussion, one can characterize the measure $\mathrm{d}W$ by its density with respect to $\mathrm{d}\mu$ and the function W decreases if its density $(\mathrm{d}W/\mathrm{d}\mu)(t) \leqslant 0$ for all $t \geqslant 0$. We recall Moreau's rule for differentiation of quadratic functions of RCLVB functions [22, pp. 8–9]: let $u(\cdot)$ be RCLBV, then $\mathrm{d}(u^2) = (u^+ + u^-)\,\mathrm{d}u$ where u^+ and u^- are the right-limit and left-limit functions of $u(\cdot)$. Let us now compute the density of the measure $\mathrm{d}W$ with respect to $\mathrm{d}\mu$:

$$\frac{\mathrm{d}W}{\mathrm{d}\mu}(t) = \tfrac{1}{2}[\dot{q}(t^+) + \dot{q}(t^-)]^{\mathrm{T}}M(q(t))\,\frac{\mathrm{d}v}{\mathrm{d}\mu}(t)$$

$$+ \frac{\partial U}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}\mu}(t) + \frac{1}{2}\frac{\partial}{\partial q}(\dot{q}(t^+)^{\mathrm{T}}M(q(t))\dot{q}(t^+))$$

$$\times \frac{\mathrm{d}q}{\mathrm{d}\mu}(t),$$
(13)

where $\mathrm{d}q = v(t)\,\mathrm{d}t$ since the function $v(\cdot)$ is Lebesgue integrable.

Let us now choose $\mathrm{d}\mu = \mathrm{d}t + \mathrm{d}\mu_a$. Since $(\mathrm{d}t/\mathrm{d}\mu)(t_k) = 0$ and $(\mathrm{d}q/\mathrm{d}\mu)(t_k) = 0$ whereas $(\mathrm{d}v/\mathrm{d}\mu)(t_k) = v(t_k^+) - v(t_k^-) = \dot{q}(t_k^+) - \dot{q}(t_k^-)$, it follows from (13) that at impact times one gets

$$\frac{\mathrm{d}W}{\mathrm{d}\mu}(t_k) = \tfrac{1}{2}[\dot{q}(t_k^+) + \dot{q}(t_k^-)]^{\mathrm{T}}M(q(t))[\dot{q}(t_k^+) - \dot{q}(t_k^-)]$$

$$= T_L(t_k) \leqslant 0,$$
(14)

where $T_L(t_k)$ is in (9). Let the matrix function $\dot{M}(q, \dot{q})$ be defined by $\dot{M}(q(t), \dot{q}(t)) = (\mathrm{d}/\mathrm{d}t)M(q(t))$. Let us use the expression of $F(q, \dot{q})$ given after (8), and let us assume that Christoffel's symbols of the first kind are used to express the vector $C(q, \dot{q})\dot{q} = \dot{M}(q, \dot{q}) - \tfrac{1}{2}[(\partial/\partial q)(\dot{q}^{\mathrm{T}}M(q(t))\dot{q})]^{\mathrm{T}}$. Then the matrix $\dot{M}(q, \dot{q}) - 2C(q, \dot{q})$ is skew-symmetric [31, Section 1.2]. Now if $t \neq t_k$, one gets $(\mathrm{d}v/\mathrm{d}\mu)(t) = \dot{v}(t) = \ddot{q}(t)$ and $(\mathrm{d}t/\mathrm{d}\mu)(t) = 1$ [22, p. 76] and one can calculate from (13), using the dynamics and the skew-symmetry property:

$$\frac{\mathrm{d}W}{\mathrm{d}\mu} = \frac{\mathrm{d}W}{\mathrm{d}t} = -\dot{q}^{\mathrm{T}}C(q, \dot{q})\dot{q} + \tfrac{1}{2}\dot{q}^{\mathrm{T}}\dot{M}(q, \dot{q})\dot{q} - \dot{q}^{\mathrm{T}}z_1$$

$$= -\dot{q}^{\mathrm{T}}z_1,$$
(15)

where $z_1 \in -\partial\psi_{V(q(t))}(w(t))$ and $W(\cdot)$ is defined in (12). To simplify the notation we have dropped arguments in (15), however $\dot{q}$ is to be understood as $\dot{q}(t) = \dot{q}(t^+)$ since $t \neq t_k$. Now since for all $t \geqslant 0$ $\dot{q}(t^+) \in V(q)$ [24] which is polar to $\partial\psi_\Phi(q(t))$, and from the inclusion in Appendix A it follows that $z_1^{\mathrm{T}}\dot{q}(t^+) \geqslant 0$. Therefore the measure $\mathrm{d}W$ is non-positive. Consequently the *function* $W(\cdot)$ is non-increasing [10, p. 101] and Lemma 3 is proved. $\square$

Remark 6.

- The inclusion of the indicator function $\psi_\Phi(q(t))$ in the Lyapunov function not only guarantees its positive definiteness (which anyway is assured along solutions of (11), which remain in Φ), but it also allows one to consider cases where the smooth potential has a minimum that is outside Φ. Saying "$\psi_\Phi(q) + U(q)$ has a strict minimum at q^*" is the same as saying "$U(q)$ has a strict minimum at q^* inside Φ". Since the indicator function has originally
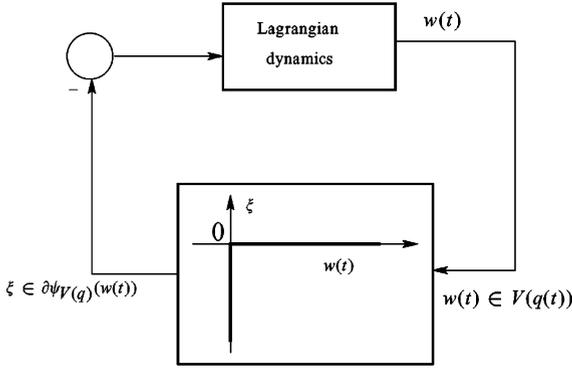
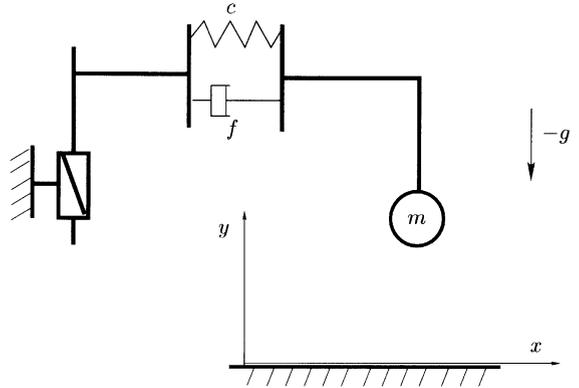Fig. 6. Unilaterally constrained Lagrangian system.

Fig. 7. A simple 2-D of system.

been introduced by Moreau as a potential associated to unilateral constraints, it finds here its natural use. In fact, we could have kept the indicator function in the stability analysis. This would just add a null term $\dot{q}(t^+)^{\mathrm{T}} z_2 (\mathrm{d}t/\mathrm{d}\mu)(t)$ in the right-hand side of (13), with $z_2 \in \partial\psi_\Phi(q(t))$.

- As alluded to above, taking $e = 1$ in (8) assures that there is no accumulation of impacts, thus the sequence of impact times $\{t_k\}_{k \geqslant 0}$ can be ordered, $\mathrm{d}\mu_a = \sum_{k \geqslant 0} \delta_{t_k}$, and velocities are piecewise continuous. Then a much simpler formulation can be adopted by separating continuous motion phases occurring on intervals (t_k, t_{k+1}) from impact times. The system is therefore non-Zeno for $e = 1$ and if Assumption 1 holds.
- One does not need to make further assumptions on the measure $\mathrm{d}\mu_a$ to conclude, and one sees that this conclusion is obtained directly applying general differentiation rules of RCLBV functions. The dynamics might even contain dense sets of velocity discontinuities, (13) and (14) would continue to hold. This shows that using the MDI formalism in (10) or (11) places the stability analysis in a much more general perspective than, say, restricting $\dot{q}(\cdot)$ to be piecewise continuous.

Let us note that the dynamics in (10) has the interpretation as in Fig. 6, where $\xi \in \partial\psi_{V(q(t))}(w(t))$. Since $\partial\psi_{V(q(t))}(w(t)) \subseteq N_\Phi(q) = V^\star(q)$ (the cone polar to $V(q)$ [27]), the feedback loop in Fig. 6 contains the cone complementarity problem [9]

$$N_\Phi(q) \supseteq \partial\psi_{V(q(t))}(w(t)) \ni \xi \perp w(t) \in V(q). \qquad (16)$$

When $m = 1$ and $q \in \partial\Phi$, one has $V(q) = \mathbb{R}^+$ and $N_\Phi(q) = \mathbb{R}^-$ in a suitable frame attached to q, and the graph of the multivalued mapping is the so-called corner law. In general, this is an example of an m-dimensional monotone multivalued mapping $w(t) \mapsto \xi$. Thus, Lemma 3 extends the absolute stability problem studied in the foregoing section towards another type of dynamical systems. It is noteworthy that the feedback loop in Fig. 6 contains both the complementarity conditions and the collision mapping in (8). A quite similar structure can be found for the dynamics in (6) with the state re-initialization mapping proposed in [7], which is an extension of (16).

The Lagrange–Dirichlet theorem serves as a basis for the stabilization of unconstrained mechanical systems, both in the regulation and the tracking cases [18]. In the regulation case, one may shape the potential energy so that the new equilibrium of the closed-loop system is the desired one. Lemma 3 can be used to that aim by shaping the potential $\psi_\Phi(q) + U(q)$ to a new closed-loop potential $\psi_\Phi(q) + \tilde{U}(\tilde{q})$ via a PD control where $\tilde{q} = q - q_\mathrm{d}$ and q_d is some constant desired position. However the closed-loop fixed points have to satisfy the generalized equation $0 \in \partial\psi_\Phi(q^*) + (\partial\tilde{U}/\partial\tilde{q})(\tilde{q}^*)$ with $\tilde{q}^* = q^* - q_\mathrm{d}$, which reflects the fact that $q^* \in \Phi$. Extension towards tracking is more complex but can be done [4,6].

Example 1. Let us consider the system in Fig. 7 whose dynamics is given by

$$m\ddot{x} = -f\dot{x} - cx,$$

$$m\ddot{y} = -mg + \lambda, \quad 0 \leqslant y \perp \lambda \geqslant 0,$$

$$\dot{y}(t_k^+) = -e\dot{y}(t_k^-) \quad \text{when } y(t_k) = 0. \tag{17}$$

Obviously $(q, \dot{q}(t^+)) = (0, 0)$ is the unique fixed point of (17) and one can check that the same dynamics with $y \leqslant 0$ no longer possesses any fixed point. For any $y(0) > 0$ and $e \in (0, 1)$ the sequence of impact times has a finite accumulation t_∞ [4]. The function in (12) is equal to

$$W(q, \dot{q}) = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 + \psi_{\mathbb{R}^+}(y) + mgy$$
$$+ \tfrac{1}{2}cx^2. \tag{18}$$

One may check that this function has the properties required in the proof of Lemma 3.

5. Conclusions

This note examines the extension of the absolute stability problem and of the Lagrange–Dirichlet theorem, to the case when nonlinearities in the feedback loop are multivalued monotone mappings. Some of the considered systems may also be recast in the framework of complementarity systems, which in turn may be seen as a special class of hybrid dynamical systems. Another, fruitful point of view, is that of differential inclusions, with or without measures. This class of systems encompasses mechanical systems with unilateral constraints and impacts, as well as dissipative linear complementarity systems without feedthrough term in the output equation. The well-posedness is carefully examined before studying the stability. Roughly speaking, the first class of nonsmooth systems studied in this paper correspond to relative degree one systems, while the second class are relative degree two systems. In particular a new existence and uniqueness proof for a class of nonsmooth dissipative systems, is given. Further extensions of the results towards more complex mechanical systems with set valued force laws [12,13], seem quite interesting.

Acknowledgements

Appendix A. Proof of Moreau's inclusion

For convenience let us rewrite the following definitions for a convex set Φ:

$$N_\Phi(q) = \{z \mid z^{\mathrm{T}}\xi \leqslant 0, \ \forall \xi \in V(q)\}, \tag{A.1}$$

which is an alternative definition to the one given in Section 2, and

$$\partial\psi_{V(q)}(w) = \{z \mid z^{\mathrm{T}}(\eta - \dot{q}) \leqslant 0, \ \forall \eta \in V(q)\}. \tag{A.2}$$

Since $V(q)$ as it is defined in (3) (replace K by Φ) is a cone and since $\dot{q} \in V(q)$, one can choose $\eta = \xi + \dot{q}$ with $\xi \in V(q)$ as a particular value for η. Thus if $z \in \partial\psi_{V(q)}(w)$ one gets $z^{\mathrm{T}}\eta \leqslant z^{\mathrm{T}}\dot{q}(t^+)$ and introducing $\eta = \xi + \dot{q}$, one gets $z^{\mathrm{T}}\xi \leqslant 0$ so that $z \in N_\Phi(q)$. Therefore Moreau's inclusion in (10) is proved.

Let us note [16] that the cones are to be understood as being attached to the same origin in the inclusion (for instance in Fig. 4, one has $\partial\psi_{V(q)}(v_1) \subset N_\Phi(q)$). Moreover, some natural identifications between spaces (the dual $T_{\dot{q}}^* T_q\mathcal{Q}$ at $\dot{q}$ of the tangent space $T_q\mathcal{Q}$ at q to the configuration space $\mathcal{Q}$, and the cotangent space $T_q^*\mathcal{Q}$) have been made, thanks to the linear structure of these spaces in which the cones $\partial\psi_{V(q)}(\cdot)$ and $N_\Phi(q)$ are defined. This allows to give a meaning to the inclusion in (10). This is just a generalization of the well-known identification between the space of velocities and that of forces acting on a particle in a 3-dimensional space, which both are identified with $\mathbb{R}^3$. More details are in [2,24].

Appendix B. The MDI at impact times

Let x and z be two vectors of a linear Euclidean space E, V be a closed convex cone of E, and N be the polar cone to V. Then from Moreau's lemma of the two cones [4, Lemma D1; 27, p. 340], one has $(x - z) \in -\partial\psi_V(x) \Leftrightarrow x = \mathrm{prox}[V, z] \Leftrightarrow z - x = \mathrm{prox}[N, z]$. Times t_k are atoms of the measure $\mathrm{d}v$ in (10). Via a suitable base change, the kinetic metric at an impact time can be considered as a Euclidean metric since $q(\cdot)$ is continuous at t_k, and in particular all the identifications between various dual spaces can be done. One gets from (10):
$$-M(q(t_k))[\dot{q}(t_k^+) - \dot{q}(t_k^-)] \in \partial\psi_{V(q(t_k))}(w(t_k^+)) \quad \Leftrightarrow$$
$$\dot{q}(t_k^+) + e\dot{q}(t_k^-) = \mathrm{prox}_{M(q(t_k))}[V(q(t_k)), (1+e)\dot{q}(t_k^-)] \Leftrightarrow$$

$$\dot{q}(t_k^+) + e\dot{q}(t_k^-) = (1+e)\text{prox}_{M(q(t_k))}[V(q(t_k)), \dot{q}(t_k^-)],$$
where the second equivalence is proved in [24].

References

[1] D.D. Bainov, P.S. Simeonov, Systems with Impulse Effects; Stability, Theory and Applications, Ellis Horwood Series in Mathematics and its Applications, Wiley, New York, 1989.

[2] P. Ballard, Formulation and well-posedness of the dynamics of rigid-body systems with perfect unilateral constraints, Phil. Trans. Royal Soc., Mathematical, Physical and Engineering Sciences, special issue on Nonsmooth Mechanics, Ser. A 359 (1789) (2001) 2327–2346.

[3] H. Brezis, Opérateurs Maximaux Monotones, North-Holland Mathematics Studies, North-Holland, Amsterdam, 1973.

[4] B. Brogliato, Nonsmooth Mechanics, Springer Communications and Control Engineering Series, Second Edition, Springer, London, 1999. Erratum and addenda available at http://www.inrialpes.fr/bip/people/brogliato/brogli.html

[5] B. Brogliato, On the control of nonsmooth complementarity dynamical systems, Phil. Trans. Royal Soc., Mathematical, Physical and Engineering Sciences, special issue on Nonsmooth Mechanics, Ser. A 359 (1789) (2001) 2369–2383.

[6] B. Brogliato, S.I. Niculescu, M. Monteiro-Marques, On tracking control of a class of complementary-slackness hybrid mechanical systems, Systems Control Lett. 39 (4) (2000) 255–266.

[7] M.K. Camlibel, W.P.M.H. Heemels, J.M. Schumacher, On linear passive complementarity systems, European J. Control 8 (3) (2002) 220–237.

[8] F.H. Clarke, Optimization and Nonsmooth Analysis, Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley, New York, 1983.

[9] R.W. Cottle, J.S. Pang, R.E. Stone, The Linear Complementarity Problem, Academic Press, New York, 1992.

[10] J. Dieudonné, Eléments d'Analyse, tome 2, Gauthier-Villars, Paris, 1969.

[11] E. Dimnet, M. Frémond, Collisions involving solids and fluids, in: M. Frémond, F. Macceri (Eds.), Novel Approaches in Civil Engineering, Springer, Berlin, 2001.

[12] M. Frémond, Non-Smooth Thermomechanics, Springer, Berlin, 2002.

[13] Ch. Glocker, Set-valued Force Laws, Springer Lecture Notes in Applied Mechanics, Vol. 1, Springer, Berlin, 2001.

[14] D. Goeleven, D. Motreanu, V.V. Motreanu, On the stability of stationary solutions of parabolic variational inequalities, Adv. Nonlinear Variational Inequalities 6 (2003) 1–30.

[15] D. Goeleven, G.E. Stavroulakis, G. Salmon, P.D. Panagiotopoulos, Solvability theory and projection method for a class of singular variational inequalities: elastostatic unilateral contact applications, J. Optim. Theory Appl. 95 (2) (1997) 263–293.

[16] M. Kunze, M.D.P. Monteiro Marques, An introduction to Moreau's sweeping process, in: B. Brogliato (Ed.), Impacts in Mechanical Systems, Analysis and Modelling, Springer Lecture Notes in Physics, Vol. 551, Springer, Berlin, 2000, pp. 1–60.

[17] Y.J. Lootsma, A.J. van der Schaft, M.K. Camlibel, Uniqueness of solutions of linear relay systems, Automatica 35 (3) (1999) 467–478.

[18] R. Lozano, B. Brogliato, O. Egeland, B. Maschke, Dissipative Systems Analysis and Control, Springer Communications and Control Engineering Series, Springer, London, 2000.

[19] M. Mabrouk, A unified variational model for the dynamics of perfect unilateral constraints, European J. Mech. A/Solids 17 (5) (1998) 819–842.

[20] A. Megretski, A. Rantzer, System analysis via integral quadratic constraints, IEEE Trans. Automat. Control 42 (6) (1997) 819–830.

[21] A.N. Michel, K. Wang, B. Hu, Qualitative Theory of Dynamical Systems, Second Edition, Marcel Dekker, New York, 2001.

[22] M.D.P. Monteiro-Marques, Differential Inclusions in Nonsmooth Mechanical Problems: Shocks and Dry Friction, in: PNLDE, Vol. 9, Birkhäuser, Boston, 1993.

[23] J.J. Moreau, Fonctionnelles Convexes, Séminaire sur les équations aux dérivées partielles, Collège de France, Paris, 1966–1967.

[24] J.J. Moreau, Unilateral contact and dry friction in finite freedom dynamics, in: J.J. Moreau, P.D. Panagiotopoulos (Eds.), Nonsmooth Mechanics and Applications, CISM Courses and Lectures, Vol. 302, International Centre for Mechanical Sciences, Springer, Berlin, 1988, pp. 1–82.

[25] J.J. Moreau, M. Valadier, A chain rule involving vector functions of bounded variation, J. Funct. Anal. 74 (1986) 333–345.

[26] P.D. Panagiotopoulos, Inequality Problems in Mechanics and Applications, Convex and Non-convex Energy Functions, Birkhäuser, Boston, 1985.

[27] R.T. Rockafellar, Convex Analysis, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1970.

[28] W. Rudin, Analyse Réelle et Complexe, Dunod, Paris, 1998.

[29] A. van der Schaft, H. Schumacher, An Introduction to Hybrid Dynamical Systems, Springer Lecture Notes in Control and Information Science, Vol. 251, Springer, London, 2000.

[30] M. Vidyasagar, Nonlinear Systems Analysis, Second Edition, Prentice-Hall, Englewood Cliffs, NJ, 1993.

[31] C. Canudas de Wit, B. Siciliano, G. Bastin (Eds.), Zodiac, Theory of Robot Control, Communications and Control Engineering Series, Springer, London, 1996.