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Some results on the controllability of planar variational inequalities

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Abstract

This note deals with the controllability of a class of planar complementarity dynamical systems, which can also be viewed as planar evolution variational inequalities. It is shown that the complementarity conditions influence the controllability of the system a lot.

Keywords: Variational inequalities; Complementarity systems; Projected dynamics; Unilateral dynamics; Controllability

1. Introduction

Hybrid dynamical systems constitute a very large class of systems [13]. It is consequently necessary to focus on specific subclasses to make their study possible, see e.g. [1] for controllability issues in piecewise-linear systems. An interesting subclass is made of the so-called complementarity systems [2,8]. Similar to the fact that the stability of unilaterally constrained systems can significantly differ from that of their unconstrained counterpart [6,11], it will be shown that their controllability properties can differ a lot as well. This reinforces the fact that such nonsmooth dynamical systems deserve full attention and are not a mere extension of unconstrained or bilaterally constrained systems. In this note, we will restrict ourselves to a narrow class of complementarity-systems, that we call planar evolution variational inequalities. These systems are also sometimes called projected dynamical systems [4,11] and are used to model the dynamics of oligopolistic markets, spatial price equilibrium, elastic demand traffic equilibrium [11]. As illustrated at the end of the note, they can also model some electrical circuits with ideal diodes. In this note it is shown that the controllability of such systems depends a lot on the convex set within which the state is constrained to evolve.

2. Planar evolution variational inequalities

The linear complementarity systems (LCS) [8] we are dealing with in this study, possess the following

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dynamics:

$$\begin{aligned}&\dot{z}_1(t)=z_2(t)+C_1^{\mathrm{T}}\lambda,\\&\dot{z}_2(t)=u(t)+C_2^{\mathrm{T}}\lambda,\\&0\leqslant\lambda\perp Cz(t)+d\geqslant 0,\end{aligned}\tag{1}$$

where $C=(C_1C_2)\in\mathbb{R}^{m\times 2}$, $C_1\in\mathbb{R}^m$ and $C_2\in\mathbb{R}^m$ are the two columns of C, $d\in\mathbb{R}^m$, $\lambda\in\mathbb{R}^m$. The non-negativity is understood componentwise and $m<+\infty$. The LCS in (1) is equivalent to the linear evolution variational inequality (VI)

$$\begin{aligned}&\langle\dot{z}(t)-Az(t)-Bu(t),v-z(t)\rangle\geqslant 0,\quad\forall v\in K,\\&z(t)\in K,\quad\forall t\geqslant 0,\end{aligned}\tag{2}$$

where $z=(z_1,z_2)^{\mathrm{T}}\in\mathbb{R}^2$, $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $B=\begin{pmatrix}0\\1\end{pmatrix}$, $K=\{z|Cz+d\geqslant 0\}=\{(z_1,z_2)\in\mathbb{R}^2|C_1z_1+C_2z_2+d\geqslant 0\}$. The equivalence between (1) and (2) is obtained by noting that

$$\begin{aligned}&\begin{cases}\dot{z}(t)=Az(t)+Bu(t)+C^{\mathrm{T}}\lambda\\ \quad 0\leqslant\lambda\perp Cz(t)+d\geqslant 0\end{cases}\\&\Leftrightarrow\ -\dot{z}(t)+Az(t)+Bu(t)\in C^{\mathrm{T}}\partial\psi_{(\mathbb{R}^+)^m}(Cz(t)+d)\\&\Leftrightarrow\ \begin{cases}-\dot{z}(t)+Az(t)+Bu(t)\in N_K(z(t))\\ \qquad z(t)\in K,\quad\forall t\geqslant 0,\end{cases}\end{aligned}\tag{3}$$

where $\psi_{(\mathbb{R}^+)^m}(\cdot)$ is the indicator function of $(\mathbb{R}^+)^m$, $\partial\psi_{(\mathbb{R}^+)^m}(\cdot)$ is its subdifferential, and $N_K(\cdot)$ is the normal cone to K. The last formalism in (3) is exactly (2), from the definition of the normal cone. These equivalences are obtained from standard convex analysis rules and definitions [10] and are not developed further here for the sake of brevity.

It is noteworthy that, seen from an LCS point of view, the controlled dynamics in (1) is rather a narrow class. However, the VI formalism in (2) shows that it is not so restrictive from an application point of view, since VI are widely used in some domains of science (see [11] for market and finance applications). VI can also represent some electrical circuits with ideal diodes [6]. The LCS in (1) is a particular gradient complementarity system [9], which is in turn equivalent to the so-called projected dynamical systems [4,11,15]. There certainly remains a big gap between this work, and obtaining similar results for general projected dynamical systems (to say nothing for general LCS). Since the studies on controllability of this type of dynamical systems are rare, this paper nevertheless has some interest. The following lemma is a direct consequence of [7, Corollary 2.2]:

Lemma 1. *Consider the system in* (1). *For all continuous and locally differentiable inputs* $u(\cdot)$, *a continuous and right-differentiable solution with locally bounded derivative exists and is unique on* $[0,+\infty)$.

Let us now introduce a controllability definition.

Definition 1. The system in (1) (equivalently in (2)) is said to be K-controllable, if any state $z_f\in K$ can be reached from any state $z_i\in K$, in a finite or infinite time T, and with an admissible input $u(\cdot)$.

Admissibility of the input means that the well-posedness conditions of Lemma 1 are respected. We do not make the difference between finite and infinite T to simplify the presentation (as we shall see below, this allows us to consider the controllability in the whole of the closed convex set K without excluding some isolated points of the boundary ∂K).

The objective of this work is to prove that, under some restrictions on the convex set K, K-controllability holds. To begin with and to motivate the study, let us remark that in case $m=1$ and $K=\{z\,|\,z_2\geqslant -c, c<0\}$, then surely the system is not K-controllable. Indeed z_1 can only move from the left to the right in the phase plane, since $\dot{z}_1=z_2\geqslant -c>0$. This controlled VI is accessible [12] with reachable subspaces from $(z_1(0),z_2(0))$ equal to $\{(z_1,z_2)\,|\,z_1\geqslant z_1(0), z_2\geqslant -c\}$, but not K-controllable.

Let us note that adding some "imaginary" state re-initialization rules on ∂K such that K-controllability holds, is not envisaged here since the dynamical systems in (1) or (2) are the topic of the study. However motivated by this simple example of non-controllability, one guesses that a crucial step in the study will be to prove whether or not one is able to move on ∂K in order to reach some states which are otherwise unreachable. Due to the complementarity conditions (third line in (1)) which imply that the vector field is modified when ∂K is attained, this will under certain conditions be possible.

3. Main result

The following assumption is made and supposed to hold in the sequel:

Assumption 1. *The set K has a positive measure in* $\mathbb{R}^2$.

It is easy to construct C and d in (1) such that indeed $K=\emptyset$ or it has zero measure. Polyhedra with a positive area are an example of sets K, as well as cones (see Fig. 1), or half-planes.

Let $C_1=(a_1,\dots,a_m)^{\mathrm{T}},\ C_2=(b_1,\dots,b_m)^{\mathrm{T}},\ d=(d_1,\dots,d_m)^{\mathrm{T}}$ and let us denote the faces of the convex set K as D^i, such that $D^i\subseteq\{z|a_iz_1+b_iz_2+d_i=0\}$ and $\bar{D}^i=\{z|a_iz_1+b_iz_2+d_i=0\}$. In other words the faces are segments D^i (possibly unbounded, like in the case K is a cone, or if K is defined as a half-space), and the segments can be extended to straight lines $\bar{D}^i$ whose equations in the plane are $a_iz_1+b_iz_2+d_i=0,\ 1\leqslant i\leqslant m$. For instance in Fig. 1 and considering the set K_1, one has $D^1=A'A$ whereas $\bar{D}^1$ is the line passing through A' and A and intersecting $\{z|z_2=0\}$ at B. Let us place ourselves in the phase plane of the system, with the two axis $(0,z_1)$ and $(0,z_2)$.

Then the following is true.

Proposition 1. *The system in* (1) (*equivalently in* (2)) *is K-controllable, if and only if, there is no face of K such that*:

- *there is a portion of* D^i *with finite negative slope on the right* (*resp. left*) *of the point* $\bar{D}^i\cap\{z|z_1=0\}$, *when K is below* (*resp. above*) D^i;
- D^i *is vertical and above* (*resp. below*) $\{z|z_2=0\}$ *if K is on the right* (*resp. left*) *of* D^i;
- D^i *is horizontal and in the half-space* $\{z|z_2<0\}$ (*resp.* $\{z|z_2>0\}$) *if K is below* (*resp. above*) D^i;
- $D^i=\{z|z_2=0\}$.

For instance in Fig. 1, the faces $A'A$ of K_1, or DC of K_4, preclude controllability because they satisfy the first item.

Let us state intermediate results which characterize the motion on the boundary ∂K (Fig. 2). The proof of Proposition 1, will then be a direct consequence of Lemma 2. In the next lemma, we place ourselves in the case when there is a single constraint and we study the behavior of the system on this constraint. When K has several faces it will suffice to consider each of them separately and apply the results of the lemma independently to each constraint. Let us consider the system in (1) or (2), with $C_1=a\in\mathbb{R},\ C_2=b\in\mathbb{R},\ d=c\in\mathbb{R}$. Let us define the coordinate change

$$x_1=bz_1-az_2+\frac{bc}{a},$$
$$x_2=az_1+bz_2+c.$$

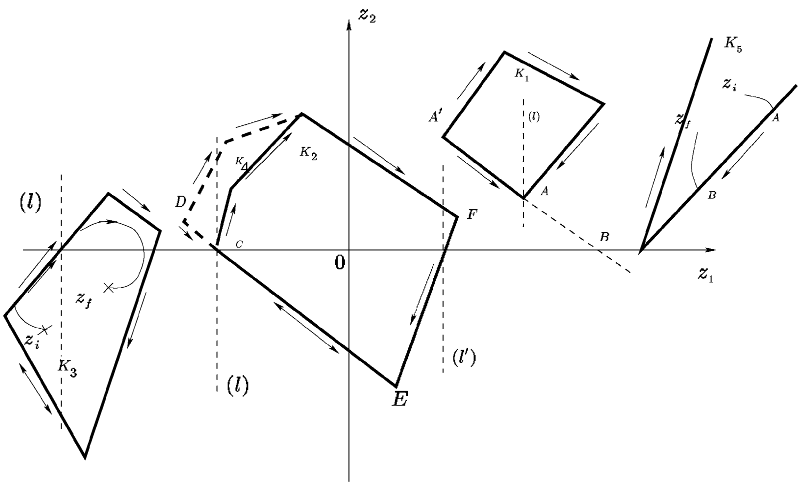

Fig. 1. Examples of K-controllable and K-uncontrollable systems.

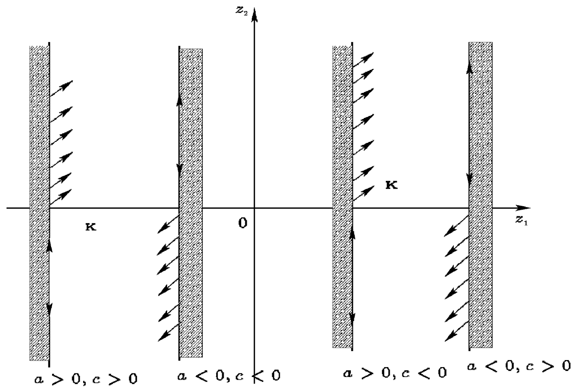

Fig. 2. Trajectories on ∂K (vertical faces).

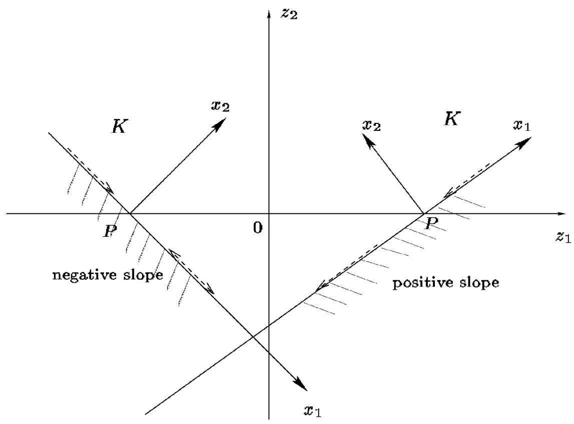

Fig. 3. The new coordinate frame.

We denote as P the intersection between the line $az_1 + bz_2 + c = 0$ and the z_1-axis, i.e., P is the origin of the new frame (x_1, x_2) and the constraint boundary is $\{x|x_2 = 0\}$. The new coordinate frame (x_1, x_2) is depicted in Fig. 3.

Then the following holds:

Lemma 2.

(i)

- *Positive slope. If $-a/b > 0,\ b \neq 0$, any point x_{1f} on the constraint can be attained from any point $x_{1i} \geqslant x_{1j}$.*
- *Negative slope. If $-a/b < 0,\ b \neq 0$, then any point x_{1f} on the constraint can be attained from any point $x_{1i} \leqslant x_{1f}$, and any point x_{1f} on the constraint can be attained from any point $x_{1i} \geqslant x_{1f}$ only on the axis $x_1 \in (P, +\infty)$. Moreover the point P can be attained from any $x_{1i} > 0$ only asymptotically.*

(ii) *If $a = 0$ then the boundary is a horizontal line $z_2 = -c/b$ and*

- *If ($b > 0$ and $c < 0$) or ($b < 0$ and $c > 0$), trajectories move from the left to the right.*
- *If ($b > 0$ and $c > 0$) or ($b < 0$ and $c < 0$), trajectories move from the right to the left.*
- *If $c = 0$ then the system remains stuck on ∂K at the contacting point.*

(iii) *If $b = 0$ then the boundary is a vertical line $z_1 = -c/a$ and*

- *If ($a > 0$ and $c > 0$) or ($a > 0$ and $c < 0$) then the system is controllable in the set $\{z|z_2 < 0\}$ and any trajectory initialized in the set $\{z|z_2 \geqslant 0\}$ detaches from ∂K.*
- *If ($a < 0$ and $c < 0$) or ($a < 0$ and $c > 0$) then the system is controllable in the set $\{z|z_2 > 0\}$ and any trajectory initialized in the set $\{z|z_2 \leqslant 0\}$ detaches from ∂K.*

Let us note that the case $a = b = 0$ is meaningless since, the system is no longer constrained, hence it is not treated in Lemma 2. We note that the two depicted cases can be rotated to obtain the admissible domain below the boundary. The axis (P, x_2) points inside the admissible set K. The dashed arrows on ∂K indicate the directions in which trajectories can be controlled on ∂K. In Fig. 1 the arrows also indicate the possible directions of motion on ∂K. Due to the complementarity conditions, it follows that in some regions of ∂K, trajectories are restricted to move in a single direction (otherwise they leave ∂K). The cases when the boundary is vertical, is depicted in Fig. 2.

Proof of Lemma 2. (i) It is simple to calculate that the dynamics (1) in the coordinates (x_1, x_2)

is

$$\begin{aligned}\dot{x}_1(t)&=\frac{-ab}{a^2+b^2}\,x_1(t)+\frac{b^2}{a^2+b^2}\,x_2(t)-au,\\ \dot{x}_2(t)&=\frac{-a^2}{a^2+b^2}\,x_1(t)+\frac{ab}{a^2+b^2}\,x_2(t)\\ &\quad+bu+(a^2+b^2)\lambda,\\ 0&\leqslant\lambda\perp x_2(t)\geqslant 0.\end{aligned}\tag{4}$$

Let us study the dynamics when the system evolves on $\{x|x_2=0\}$ on an interval $[\tau,\tau+\varepsilon)$, $\varepsilon>0$. Consequently, the derivatives $x_2^{(i)}=0$ as well, for all $i\geqslant 1$ on $(\tau,\tau+\varepsilon)$. Then on $[\tau,\tau+\varepsilon)$ the complementarity condition $0\leqslant\lambda\perp x_2\geqslant 0$ implies that $0\leqslant\lambda\perp\dot{x}_2\geqslant 0$. Indeed the "velocity" can point only inside K on $[\tau,\tau+\varepsilon)$. Therefore, one can replace (4) by

$$\begin{aligned}\dot{x}_1(t)&=\frac{-ab}{a^2+b^2}\,x_1(t)+\frac{b^2}{a^2+b^2}\,x_2(t)-au(t),\\ \dot{x}_2(t)&=\frac{-a^2}{a^2+b^2}\,x_1(t)+\frac{ab}{a^2+b^2}\,x_2(t)\\ &\quad+bu(t)+(a^2+b^2)\lambda,\\ 0&\leqslant\lambda\perp\dot{x}_2(t)\geqslant 0.\end{aligned}\tag{5}$$

Now since $x_2=0$ in the considered time interval one deduces from (5)

$$\begin{aligned}&\dot{x}_1(t)=\frac{-ab}{a^2+b^2}\,x_1(t)-au(t),\\ &\frac{-a^2}{a^2+b^2}\,x_1(t)+bu(t)+(a^2+b^2)\lambda=0,\\ &0\leqslant\lambda\perp\dot{x}_2(t)\geqslant 0.\end{aligned}\tag{6}$$

Since $\lambda\geqslant 0$ on ∂K, one deduces that $-a^2/(a^2+b^2)x_1+bu\leqslant 0$ is a necessary and sufficient condition that both x_1 and $u(\cdot)$ have to satisfy so that the system keeps moving on a face included in $\{x|x_2=0\}$. Detachment from this face occurs at a time $t_{\rm d}$, if and only if, $\dot{x}_2(t_{\rm d})>0$, which implies $\lambda(t_{\rm d})=0$ and $-a^2/(a^2+b^2)x_1(t_{\rm d})+bu(t_{\rm d})>0$. We can say that the system *grazes* the surface when $x_2=0$ and $\lambda=0$. We conclude that the dynamics on a face included in $\{x|x_2=0\}$ is given by

$$\begin{aligned}&\dot{x}_1(t)=-\frac{ab}{a^2+b^2}\,x_1(t)-au(t),\\ &-\frac{a^2}{a^2+b^2}\,x_1(t)+bu(t)\leqslant 0.\end{aligned}\tag{7}$$

If $a\neq 0$ then the feedback $u=-1/a(v+[ab/(a^2+b^2)]x_1)$ yields from (7)

$$\begin{aligned}&\dot{x}_1(t)=v(t),\\ &-x_1(t)-\frac{b}{a}v(t)\leqslant 0,\end{aligned}\tag{8}$$

where v is the new input. We notice that if $-x_1-(b/a)v=0$ then the system grazes ∂K. If $-a/b>0$, $b\neq 0$, then necessarily $v\leqslant-(a/b)x_1$, and v can be chosen <0 so that x_1 can be made to decrease while staying on ∂K. If $-a/b<0$, $b\neq 0$, then necessarily $v\geqslant-(a/b)x_1$. If $x_1<0$ then $v>0$, so on $(-\infty,0)$, x_1 can only increase. On $(0,+\infty)\ni x_1$, one can choose $v=-(a/b)x_1$ so that P is attained only asymptotically from any $x_{1i}>0$.

(ii) Now if $a=0$ (and consequently $b\neq 0$), the dynamics on ∂K is given by

$$\begin{aligned}&\dot{z}_1(t)=-\frac{c}{b},\\ &z_2(t)=-\frac{c}{b},\\ &u(t)+b\lambda=0\quad\text{and}\quad\lambda\geqslant 0\Rightarrow bu(t)\leqslant 0.\end{aligned}\tag{9}$$

This is obtained in a similar way as above, noting that on ∂K ones has $b\dot{z}_2=0$ and $0\leqslant\lambda\perp b\dot{z}_2\geqslant 0$. The results follow. The detachment from the surface $bz_2+c=0$ occurs, if and only if, $b\dot{z}_2(t_{\rm d})>0$ at some time $t_{\rm d}$, i.e., $bu(t_{\rm d})+b^2\lambda(t_{\rm d})=bu(t_{\rm d})>0$ (indeed $\lambda(t_{\rm d})=0$ from the complementarity conditions).

(iii) If $b=0$ (and consequently $a\neq 0$) the dynamics on ∂K is given by

$$\begin{aligned}&z_1(t)=-\frac{c}{a},\\ &\dot{z}_2(t)=u(t),\\ &z_2(t)+a\lambda=0\quad\text{and}\quad\lambda\geqslant 0\Rightarrow az_2(t)\leqslant 0.\end{aligned}\tag{10}$$

The results stated in Lemma 2 (iii) are a direct consequence of (10). □

Lemma 3. *The unilateral constraint* $\{x|x_2\geqslant 0\}$ *in* (1) *can be activated or deactivated with a continuous input signal* $u(\cdot)$.

Proof. Let us consider (1) or equivalently (4). The contact phases, or active constraint, are characterized by $\lambda\geqslant 0$ and $x_2=0$ whereas the non-contact phases, or inactive constraint, correspond to $\lambda=0$ and $x_2>0$. When steering the state inside K (i.e., in $K\backslash\partial K$) it is always possible to attain the boundary ∂K, and to remain on ∂K, with a continuous input. Indeed this amounts to finding a path in the phase plane (z_1, z_2), parameterized by t, linking two points $z^0 \in \mathrm{Int}(K)$ and $z^1 \in \partial K$, such that its second derivative with respect to t satisfies the inequality in (7) on ∂K. Detachment can also be forced with a continuous control input. Indeed one sees from (5) that as soon as ∂K is attained λ is the solution of a linear complementarity problem (LCP) with matrix $a^2 + b^2 > 0$ (a scalar in this case) and consequently depends continuously on $u(\cdot)$ (see e.g. [5, Example 4.8.15]). One can speak of a *controlled LCP* in (5) which can be controlled with a continuous input. Consequently the controllability result holds with input signals $u(\cdot)$ which are continuous and piecewise differentiable. This guarantees the existence of a unique solution of (1) or (4) as a result of Lemma 1. It is noteworthy that these results still hold if the codimension of ∂K is $\geqslant 2$ (activation or detachment at corners of ∂K). □

Proof of Proposition 1. The proof is done by observing that under the stated conditions, and from Lemmas 2 and 3, then any point in K can be steered by a continuous $u(\cdot)$ to any other point in K. Indeed if a state z_f cannot be attained from z_i via a trajectory in $K \backslash \partial K$, then a portion of path can be tracked on ∂K. Concatenating paths in the interior of K and on ∂K allows one to construct a path linking z_i to z_f. The conditions of Proposition 1 are sufficient but can also be seen to be necessary, for if one of them fails then there exist couples of states in K which cannot be joined by a controlled trajectory. □

Examples. From the results of Lemma 2, one sees that the boundary of the domain K_3 in Fig. 1 can be tracked clockwise. Consequently any point z_f on the right of the line (l) can be attained from any point z_i on the left of (l). There has to be a portion of the trajectory that evolves on ∂K_3 to reach z_f from z_i. Let us consider the set K_1 in Fig. 1. The system is not K_1-controllable because the only way to attain a point on the left of the vertical line (l) from a point on the right of (l), is to follow the boundary ∂K_1. However once the point A has been reached, it is impossible to move on ∂K_1 toward A'. The system can be steered on the line AA' only in the direction of B. Consequently all points of K_1 which are situated on the left of (l), cannot be attained from points in K_1 on the right of (l). It is noteworthy that even small-time local controllability [14] may fail. For instance two arbitrarily close states z_i and z_f in K_1, with z_i on the right of (l) and z_f on the left of (l), cannot be joined by a solution of (1) with some control $u(\cdot)$. Consider now K_2. Then trajectories can be controlled from E to C, though C is reachable in infinite time only. Assume that C is just below the axis $\{z | z_2 = 0\}$. It follows from Lemma 2 that ∂K_2 can be tracked clock-wise by applying some suitable control input. Thus, the points on the right of the vertical line (l') can be steered to anywhere in K_2 by first moving on FE. One may say that the dynamics is suitably modified on the boundary FE so that z_1 can decrease in the first quadrant. In the same way the system is K_5-controllable, but it is not K_4-controllable (the states on the left of the line (l) cannot be reached from the states in K_2). The system is K_5-controllable since as illustrated a state z_f that cannot be attained from z_i via a trajectory which remains in $K \backslash \partial K$, can be attained via a path $z_i A B z_f$.

Remark 1. As we said after Definition 1, including infinite time T in the controllability allows us to disregard some isolated points of K that may not be reachable in finite time. This is the case for the domain K_2 where the point C can be attained asymptotically only.

4. An example

Let us consider the simple electrical circuit in Fig. 4, where R is a resistor, L is an inductor, C is a capacitor, and the diode is supposed to be ideal. Its dynamics is given by

$$\begin{aligned} &\dot{z}_1(t) = z_2(t), \\ &\dot{z}_2(t) = -\frac{R}{L} z_2(t) + \frac{u(t)}{L} - \frac{1}{LC} z_1(t) - \frac{1}{L} \lambda, \\ &0 \leqslant \lambda \perp -z_2(t) \geqslant 0, \end{aligned} \tag{11}$$

where $z_1(\cdot)$ is the time integral of the current across the capacitor, $z_2(\cdot)$ is the current across the circuit and $-\lambda$ is the voltage of the diode, $u(\cdot)$ is a voltage control. One has $K = \{z | z_2 \leqslant 0\}$. One sees that this system is not controllable by simple application of Proposition 1. One may transform the system in (11) into the canonical form in (1), by applying a pre-feedback $u(z_1, z_2) = Lv(t) + (R/L)z_2 + (1/LC)z_1$. In fact the state $z_1(\cdot)$ can only decrease, or be

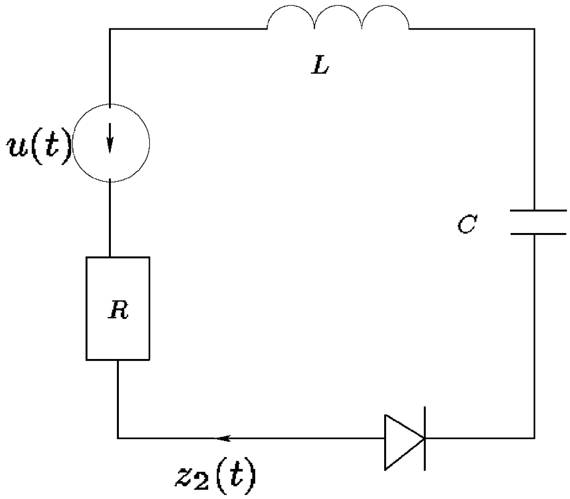

Fig. 4. A simple electrical circuit.

controlled to a constant value on ∂K. Consequently the system in (11) is not K-controllable. This is intuitively sound since it corresponds to having the capacitor loaded with a non-positive current at all times.

5. Conclusion

In this note, we have proposed a characterization of the controllability properties of planar evolution variational inequalities with control input. These systems are a subclass of complementarity dynamical systems. They are nonsmooth and nonlinear. The material in this note relies heavily on the properties of the system on the boundary of the constraint set and on the behaviour of the trajectories of planar systems in their phase plane. Consequently an extension of this work should rely on the analytical tools in [3] that characterize the control capabilities of a system, on the boundary of its admissible domain. The class of systems that is considered is a narrow class of complementarity dynamical systems. However the results show that the controllability of complementarity dynamical systems differs significantly from that of unconstrained systems.

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