1 Energy considerations

We can decompose the total energy of the system as the sum of its potential and kinetic energies, i.e., $E_t = E_p + E_c$, with

$$E_p := -\int_{\Omega} \rho \phi (x \cdot g) \quad \text{and} \quad E_c := \int_{\Omega} \frac{1}{2} \rho \phi (u \cdot u) .$$

Since we are using homogeneous Dirichlet boundary conditions, their evolution follows

$$\frac{dE_p}{dt} = -\rho \int_{\Omega} \frac{\partial \phi}{\partial t} (x \cdot g) = \rho \int_{\Omega} \nabla \cdot [\phi u] (x \cdot g)$$

and

$$\frac{dE_c}{dt} = \int_{\Omega} \rho \phi \left( \frac{D u}{D t} \cdot u \right) = \int_{\Omega} \eta \left( \nabla \cdot [\phi \dot{e}] \cdot u \right) - (\nabla \cdot [\phi \lambda] \cdot u) + \rho \phi (g \cdot u)$$

Therefore $\frac{dE_t}{dt} \leq \int_{\Omega} \phi \lambda \cdot \dot{e}$. For the system to be dissipative, it suffices that $(\lambda; \gamma) \in DP(\mu) \implies \phi \lambda : \dot{e} \leq 0$, which can be easily verified.

Indeed, let $(\lambda; \gamma) \in DP(\mu)$, we will show that

$$\begin{cases} \text{Dev} \lambda : \text{Dev} (\phi \dot{e}) \leq 0 \quad \text{(a)} \\ \text{Tr} \lambda \text{Tr} (\phi \dot{e}) \leq 0 \quad \text{(b)} \end{cases}$$

Since $\text{Dev} (\phi \dot{e}) = \text{Dev} \gamma$, (a) is trivial; either $\text{Dev} (\phi \dot{e}) = 0$, or $\text{Dev} (\phi \dot{e}) \neq 0$ and $\text{Dev} \lambda = -\left( \mu \frac{\partial \gamma}{\partial \lambda} \right)$ with $\text{Tr} \lambda \geq 0$.

Now, for (a), the complementarity condition implies either $\text{Tr} \lambda = 0$, or $\text{Tr} \lambda > 0$ and $\text{Tr} \gamma = 0$. Since $\beta \geq 0$, $\text{Tr} (\phi \dot{e}) \leq 0$, and $\text{Tr} \lambda \text{Tr} (\phi \dot{e}) \leq 0$.

2 Derivation of variational formulation

Multiplying both sides of the definition of $\gamma$ from Equation (8) by a test function $\tau$ and integrating over $\Omega$ yields

$$\int_{\Omega} \gamma : \tau = \int_{\Omega} \phi \dot{e} : \tau + \int_{\Omega} \phi_{\max} - \phi \frac{I : \tau}{3\Delta t}$$

$$= \sum V_p \left( \tau (x_p^p) \cdot \dot{e} (x_p^n) - \frac{I : \tau (x_p^n)}{3\Delta t} \right) + \int_{\Omega} \phi_{\max} I : \tau$$

$$= b(\tau, u) + k(\tau) .$$

We retrieve Equation (11), $s(\gamma, \tau) = b(\tau, u) + k(\tau)$.

We can proceed similarly for the discrete-time momentum balance equation (9),

$$\frac{\rho}{\Delta t} \phi u + \nabla \cdot [\phi (\lambda - \eta \dot{e})] = \rho \phi \left( g + \frac{u^{p\rightarrow g}}{\Delta t} \right) ,$$

and get

$$\int_{\Omega} \left( \frac{\rho}{\Delta t} \phi u \cdot \phi v \right) = m(u, v)$$

$$\int_{\Omega} \left( \rho \phi \left( g + \frac{u^{p\rightarrow g}}{\Delta t} \right) \cdot v \right) = l(v) .$$

Using the Stokes formula with our homogeneous Dirichlet boundary conditions, we also have

$$\int_{\Omega} (\nabla \cdot [\phi \lambda - \eta \phi \dot{e}] \cdot v) = \int_{\Omega} (\eta \phi \dot{e} - \phi \lambda) : D(v)$$

$$= \sum V_p \left( \eta \dot{e} (x_p^n) - \lambda (x_p^n) \right) : D(v) (x_p^n)$$

$$= a(u, v) - b(\lambda, v) .$$

and retrieve Equation (10), $m(u, v) + a(u, v) = b(\lambda, v) + l(v)$.

3 Frictional boundary condition

Suppose that $(\lambda_{\text{GB}}; \gamma_{\text{GB}}) \in DP(\mu_{\text{GB}})$, with $\gamma_{\text{GB}} := \frac{1}{3} (\bar{r}_{n_{\text{GB}}} + n_{\text{GB}} \bar{r})$, and $\| n_{\text{GB}} \| = 1$.

The force induced by a stress $\sigma$ through a plane with normal $n$ is computed as $\sigma n$; the reaction force induced by the material on the frictional boundary is therefore $\tau = \lambda_{\text{GB}} n_{\text{GB}}$. In the following, we investigate the relationship between $\tau$ and $\tau$, the relative velocity of the boundary w.r.t. the granular material.

3.1 Signorini condition

First, remark that $\text{Tr} \gamma_{\text{GB}} = (\bar{r} \cdot n_{\text{GB}}) = \bar{v}_n$.

$$r_n = \frac{1}{3} \left( \text{Tr} \lambda_{\text{GB}} + (\text{Dev} \lambda_{\text{GB}}) : (n_{\text{GB}} n_{\text{GB}}) \right)$$
and

\[
(\text{Dev} \lambda_{ab}) : (n_{ab} n_{ab}^\top) \leq 2 \left| \text{Dev} \lambda_{ab} \right| ||n_{ab} n_{ab}^\top||
\]

\[
\leq \frac{\mu_{ab}}{\sqrt{6}} \text{Tr} \lambda_{ab} \times \frac{2}{\sqrt{2}}
\]

\[
\leq \frac{\mu_{ab}}{\sqrt{3}} \text{Tr} \lambda_{ab}.
\]

Therefore,
\[
\left(1 - \frac{\sqrt{3} \mu_{ab}}{3}\right) \text{Tr} \lambda_{ab} \leq r_N \leq \left(1 + \frac{\sqrt{3} \mu_{ab}}{3}\right) \text{Tr} \lambda_{ab}.
\]

For \(\mu_{ab} < \frac{1}{\sqrt{3}}\), we thus have

\[
0 \leq \text{Tr} \gamma_{ab} \perp \text{Tr} \lambda_{ab} \geq 0 \implies 0 \leq n_{N} \perp r_N \geq 0,
\]

i.e., the Signorini condition is satisfied.

### 3.2 Tangential reaction

Notice that

\[
\text{Dev}(\gamma_{ab}) n_{ab} = \frac{1}{2} n + \frac{1}{3} (\nu \cdot n_{ab}) n_{ab}
\]

\[
= \frac{1}{2} n + \frac{2}{3} n_{N} n.
\]

**Sliding case** First suppose that \(\nu \neq 0\), therefore \(\text{Dev} \gamma_{ab} \neq 0\), and \(\text{DP}(\mu_{ab})\) imposes that \(\lambda_{ab} = -\alpha \text{Dev} \gamma_{ab}, \alpha > 0\). Since \(\nu = \frac{1}{2} \text{Tr} \lambda_{ab} n + \text{Dev} \lambda_{ab} n\), we can identify that

\[
r_{T} = \frac{1}{2} \alpha \nu_{T}
\]

i.e., the tangential friction force is opposed to the tangential relative velocity. Now, let us show that \(r_{T}\) lies on the boundary of the second-order cone of aperture \(\sqrt{\frac{3}{2}} \mu_{ab}\), i.e., \(||r_{T}|| = \sqrt{\frac{3}{2}} \mu_{ab} r_{N}||\).

From the Signorini condition, \(n_{N} > 0\) implies \(r_{N} = 0\), and therefore \(\text{Tr} \lambda_{ab} = 0\). This means \(||\text{Dev} \lambda_{ab}|| = 0\), and consequently \(||r_{T}||\).

Our relation \(||r_{T}|| = \sqrt{\frac{3}{2}} \mu_{ab} r_{N}||\) is trivially satisfied.

We now have to study the case \(\nu_{N} = 0\). \(\text{Dev} \lambda_{ab} = -\alpha \text{Dev} \gamma_{ab}\) means that

\[
||r_{T}|| = ||\text{Dev} \lambda_{ab} n|| = ||\text{Dev} \lambda_{ab} n|| \frac{||\text{Dev} \gamma_{ab} n_{ab}||}{||\text{Dev} \gamma_{ab}||}.
\]

Since \(\nu \cdot n_{ab} = 0\),

\[
||\text{Dev}(\gamma_{ab}) n_{ab}|| = \frac{1}{2} ||\nu_{T}||
\]

and

\[
||\text{Dev}(\gamma_{ab})||^2 = ||\gamma_{ab}||^2 = ||\nu_{T} n_{ab}^\top||^2 - \frac{1}{4} ||\nu_{T} n_{ab}^\top - n_{ab} \nu_{T}^\top||^2
\]

\[
= \frac{1}{2} ||\nu_{T}||^2 + \frac{1}{4} ||\nu_{T} \wedge n_{ab}||^2 = \frac{1}{2} ||\nu_{T}||^2 + \frac{1}{4} ||\nu_{T}||^2
\]

\[
= \frac{1}{4} ||\nu_{T}||^2 = ||\text{Dev}(\gamma_{ab}) n_{ab}||^2.
\]

This means

\[
||r_{T}|| = ||\text{Dev} \lambda_{ab}|| = \frac{\mu_{ab}}{\sqrt{6}} \text{Tr} \lambda_{ab} = \sqrt{\frac{3}{2}} \mu_{ab} r_{N}.
\]

The sliding case therefore satisfies the Coulomb law with coefficient \(\sqrt{\frac{3}{2}} \mu_{ab}\).

**Sticking** When \(\nu = 0\), we cannot conclude without more information about the relationship between \(\lambda_{ab}\) and \(\gamma_{ab}\). Indeed, we can only verify that

\[
||r_{T}|| \leq ||\text{Dev} \lambda_{ab} n|| \leq \sqrt{2} ||\text{Dev} \lambda_{ab}||
\]

\[
\leq \frac{\mu_{ab}}{\sqrt{3}} \text{Tr} \lambda_{ab} \leq \frac{\sqrt{3} \mu_{ab}}{1 - \sqrt{\frac{3}{2}} \mu_{ab}} r_{N}
\]

i.e. the reaction force has to lie inside a second-order cone of aperture \(\frac{\sqrt{3} \mu_{ab}}{1 - \sqrt{\frac{3}{2}} \mu_{ab}}\).

This last bound does not correspond to the one derived for the sliding case (except when \(\mu_{ab} = 0\), but nevertheless models a coupling between the tangential and normal reaction forces.

### 3.3 Reverse inclusion

For any \((r; \nu) \in C^3 \left(\sqrt{\frac{3}{2}} \mu\right)\) - i.e., satisfying the 3D Coulomb law with friction coefficient \(\sqrt{\frac{3}{2}} \mu\) - we can construct a symmetric tensor \(\lambda_{ab}\) such that \((\lambda_{ab}; \gamma_{ab}) \in \text{DP}(\mu)\). Indeed, let

\[
\lambda_{ab} := (r_{T} n_{ab}^\top + n_{ab} r_{N}^\top) + r_{N} I
\]

We have

\[
\text{Tr} \lambda_{ab} = 3 r_{N} = \sqrt{\frac{3}{2}} r_{N}
\]

\[
\lambda_{ab} n_{ab} = r_{T} + n_{ab} (r_{N}) = r
\]

\[
||\text{Dev}(\lambda_{ab})|| = ||r_{T} n_{ab}^\top + n_{ab} (r - r_{N} n_{ab})^\top|| = ||r_{T}||
\]

It can be easily verified that for any case of the \(C^3 \left(\sqrt{\frac{3}{2}} \mu\right)\) disjunctive formulation satisfied by \(r\) and \(\nu\), the corresponding case of \(\text{DP}(\mu)\) is satisfied by \((\lambda_{ab}; \gamma_{ab})\).