Local analysis of dynamical systems

and application to nonlinear waves

Guillaume JAMES

INRIA Grenoble Rhône-Alpes, Tripop Team

Univ. Grenoble Alpes, Grenoble INP, CNRS, LJK

Ecole thématique – DYNOLIN – 2018

Méthodes de dynamique non linéaire pour l’ingénierie des structures

IV – Modulation equations for lattices with strongly nonlinear spatial coupling

Outline :

- Different types of strongly nonlinear lattices and localized waves relevant to granular metamaterials

- Discrete p-Schrödinger (DpS) limit in Newton's cradle, existence of stationary breathers

  G.J. (2011)
  B. Bidégaray-Fesquet, E. Dumas, G.J. (2013)
  G.J., P. Kevrekidis, J. Cuevas (2013)
  G.J., Y. Starosvetsky (2014)

- DpS limit in mass-with-mass systems, long-lived breathers

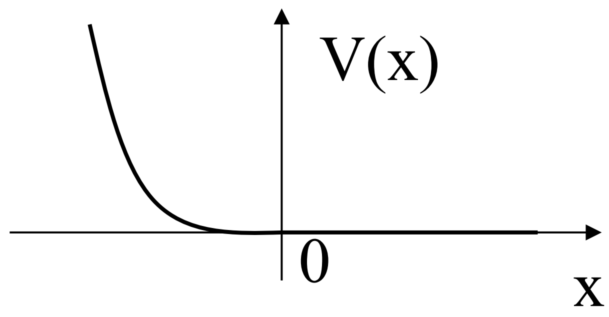
  L. Liu, G.J., P. Kevrekidis, A. Vainchtein (2016)

- Continuum limits of DpS and traveling breathers : G.J. (2018)

I – Strongly nonlinear lattices and granular metamaterials

Model 1 : Fermi-Pasta-Ulam lattice (FPU), fully nonlinear potential

$$\ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1})$$

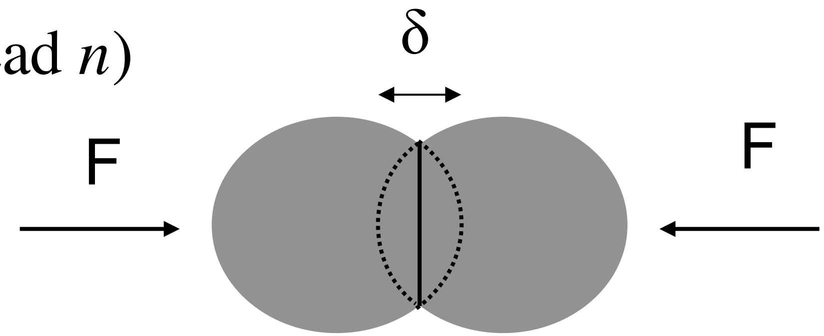


$$V(x) = \frac{1}{p}(-x)_+^p , \quad p > 2, \quad (a)_+ = \mathrm{Max}(a,0)$$

Hertz potential for p=5/2 :

contact force between two spherical beads : $\mathrm{F} \approx \delta^{3/2}$

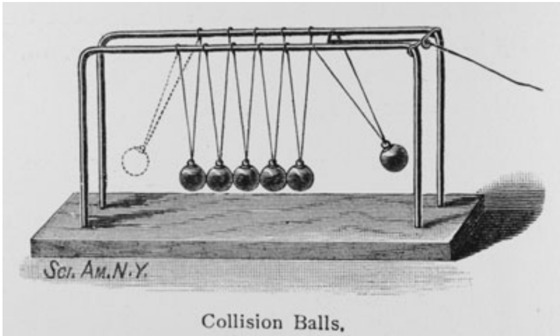
granular chain : (x_n = displacement of bead n)



Model 2 : granular chain with local potential (Newton's cradle)

$$\ddot{x}_n + \omega^2 x_n = (x_{n-1} - x_n)_+^{3/2} - (x_n - x_{n+1})_+^{3/2}$$

Classical Newton's cradle :

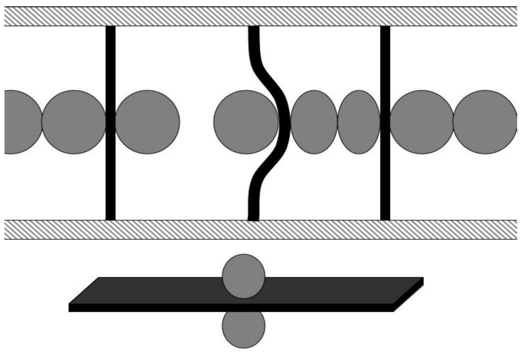


$$\Rightarrow \text{but } \omega \sim \frac{\text{bead collision time}}{\text{local oscillation period}} \ll 1$$

$$\omega \sim 10^{-4} \text{ for impact velocity } \approx 1\text{m/s}$$

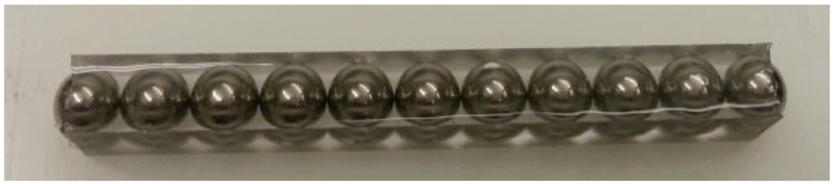
Stiff attachements (plates) : ω~1

G.J., Kevrekidis, Cuevas '13



Beads in an elastic matrix :

Hasan et al, Granular Matter '15



Model 3 : locally resonant granular chain

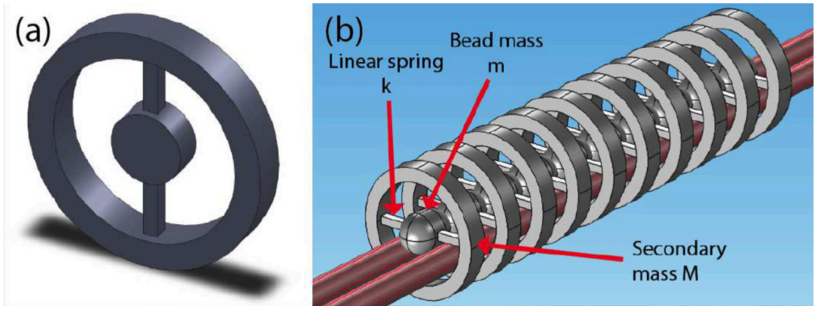
$$\ddot{x}_n + k(x_n - y_n) = (x_{n-1} - x_n)_+^{3/2} - (x_n - x_{n+1})_+^{3/2}$$

$$\rho\, \ddot{y}_n + k(y_n - x_n) = 0$$

(MwM)

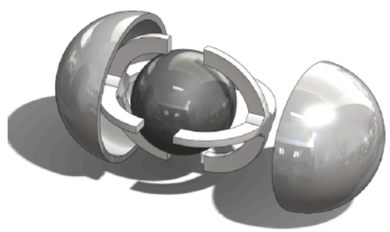
y_n: displacement of external ring resonator

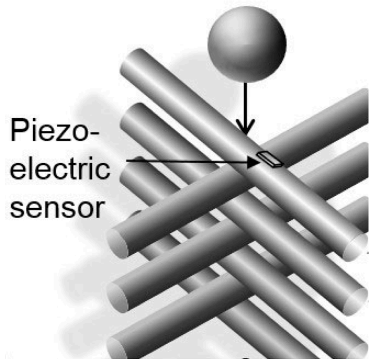
Gantzounis et al, J. Appl. Phys. 114 (2013)



Variants :

- mass-in-mass system:
  Bonanomi, Theocharis, Daraio,
  Phys Rev E 91 (2015)
- woodpile structures:
  Kim et al, PRL 114 (2015)





MwM interpolates between FPU-Hertz ($\rho = 0$) and Newton's cradle ($\rho = \infty$)

Model 4 : discrete p-Schrödinger equation (DpS)

G.J., Math. Models Meth. Appl. Sci. 21 (2011), Starosvetsky et al ‘12 (coupled chains)

$$i\,\partial_\tau A_n = (A_{n+1} - A_n)\,|A_{n+1} - A_n|^{p-2} - (A_n - A_{n-1})\,|A_n - A_{n-1}|^{p-2}$$

$p > 2,$ Hamiltonian $\sum_{n=-\infty}^{+\infty} |A_{n+1} - A_n|^p$

Reminiscent of DNLS equation, but purely intersite nonlinearity

Continuum limit : $i\,\partial_\tau A = \partial_\xi(\partial_\xi A\,|\partial_\xi A|^{p-2}) \rightarrow p$ - Laplacian

DpS equation (p=5/2 for Hertz contact) approximates the slow modulation in time of small oscillations in :

-Newton’s cradle : Bidégaray-Fesquet, Dumas, G.J. ’13

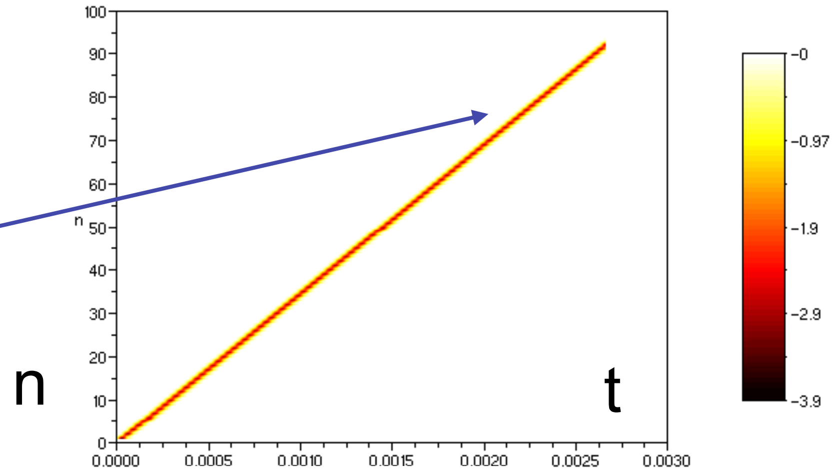
-MwM with heavy secondary masses initially close to resting state :

Liu, G.J., Kevrekidis, Vainchtein ’16

Different types of localized waves generated by localized perturbations :

FPU - Hertz :

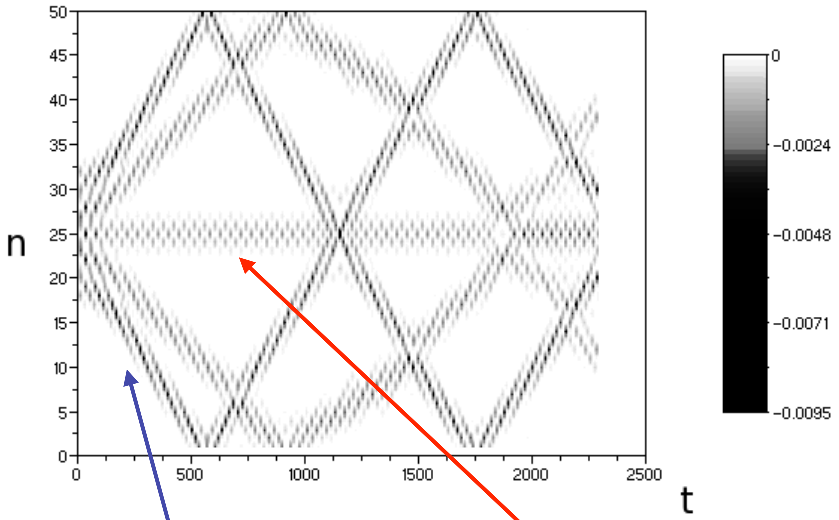
Contact forces $-(x_n - x_{n+1})_+^{3/2}$:



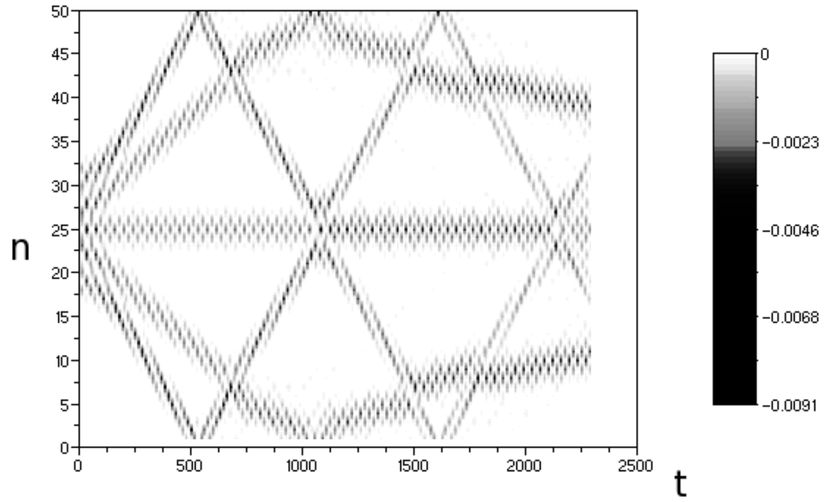
solitary wave

(Nesterenko '83)

Newton's cradle:



DpS : $x_n(t) = 2\varepsilon \,\mathrm{Re}(e^{it} A_n(\varepsilon^{1/2}\, t))$

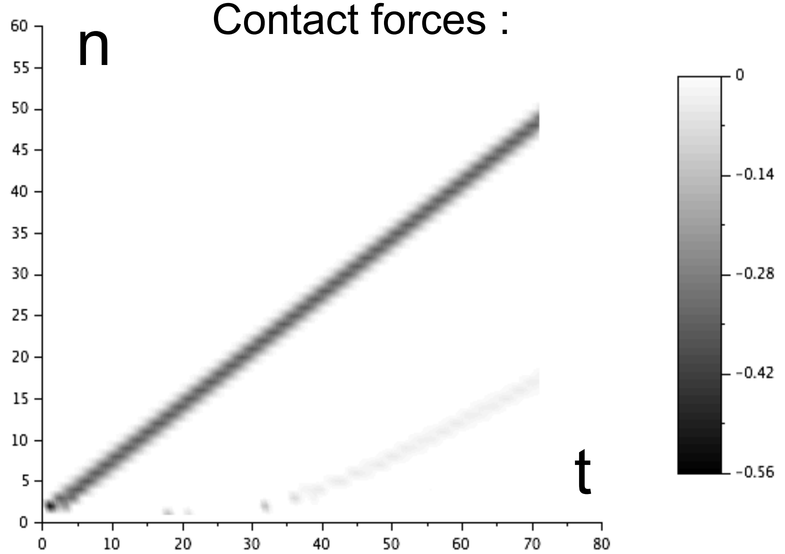


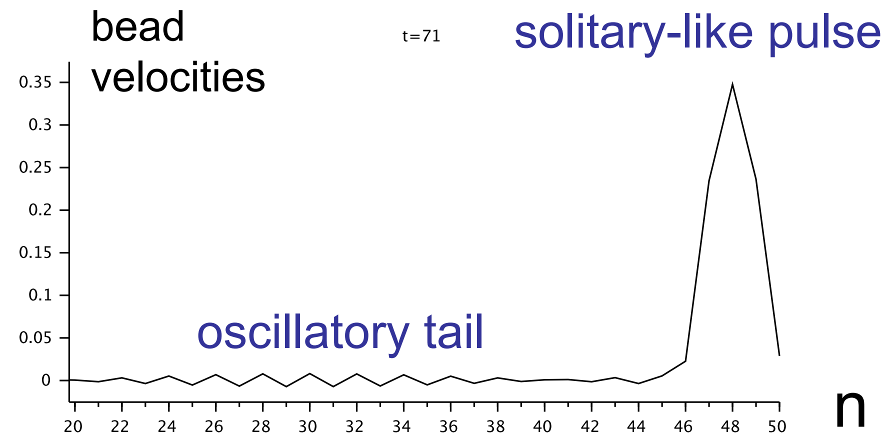
traveling breather breather (non-propagating) = localized oscillations

Different types of localized waves :

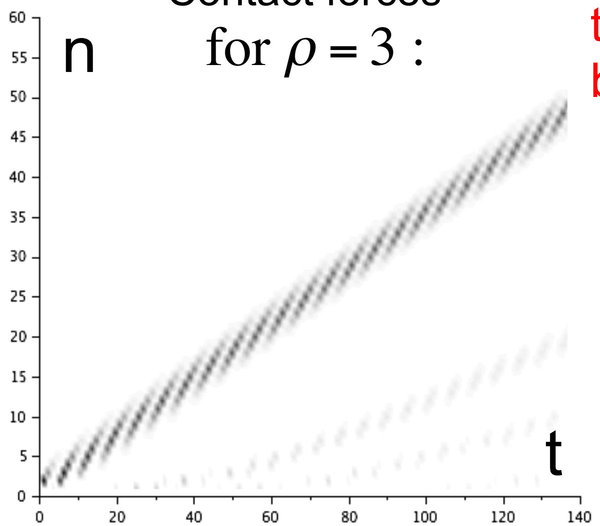
MwM for mass ratio $\rho = 1/3$:   ($k = 1,\ \dot{x}_1(0) = 1$)

Contact forces :





Contact forces
for $\rho = 3$:



traveling
breather

Study of impact propagation in MwM :

Xu, Kevrekidis, Stefanov ’15

Kim et al ’15, Vorotnikov et al ’18

II – From Newton's cradle to DpS, stationary breathers

(N.C.) $$\ddot{x}_n + x_n = (x_{n-1} - x_n)_+^{\alpha} - (x_n - x_{n+1})_+^{\alpha}$$ $(\alpha > 1)$

Leading order solutions (small amplitude ε) :

$$x_n^{A,\varepsilon}(t) = \varepsilon A_n(\varepsilon^{\alpha-1} t) e^{it} + \varepsilon \overline{A}_n(\varepsilon^{\alpha-1} t) e^{-it}$$ slow time : $\tau = \varepsilon^{\alpha-1} t$

Collect terms $O(\varepsilon^{\alpha}) \times e^{it}$ $\Rightarrow$ DpS equation (G.J. '11)

$$2\tau_0\, i\, \partial_\tau A_n = (A_{n+1} - A_n) | A_{n+1} - A_n |^{\alpha-1} - (A_n - A_{n-1}) | A_n - A_{n-1} |^{\alpha-1}$$

$(\tau_0 \approx 1.5 \text{ for } \alpha = 3/2)$

$\Rightarrow$ phase invariance, conservation of ℓ_2 norm, scale invariance

Formal derivation of the DpS equation :

$$\ddot{x}_n + x_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1})$$

$$-V'(x) = (-x)_+^{\alpha}$$

$$\alpha > 1$$

Small amplitude ε, slowly modulated time-periodic solutions :

$$x_n(t) = \varepsilon X_n(\tau, t) \qquad \tau = \varepsilon^{\alpha-1}\, t \ \text{ slow time } : \int_0^{\varepsilon^{1-\alpha}} |V'(-\varepsilon)|\, dt = \varepsilon$$

$$X_n(\tau, t + 2\pi) = X_n(\tau, t)$$

$$\left[(\varepsilon^{\alpha-1}\partial_\tau + \partial_t)^2 + 1\right] X = \varepsilon^{\alpha-1}\, \delta^+ V'(\delta^- X)$$

$$X = (X_n)_n, \qquad (\delta^+ X)_n = X_{n+1} - X_n, \qquad (\delta^- X)_n = X_n - X_{n-1}$$

Formal derivation of the DpS equation :

$$\left[(\varepsilon^{\alpha-1}\partial_\tau + \partial_t)^2 + 1\right]X = \varepsilon^{\alpha-1}\,\delta^+ V'(\delta^- X)$$

$$\text{Expansion : } X = X^0 + \varepsilon^{\alpha-1}X^1 + o(\varepsilon^{\alpha-1})$$

$$\Rightarrow \text{order } \varepsilon^0 : \quad \left[\partial_t^2 + 1\right]X^0 = 0 \Rightarrow X_n^0(\tau,t) = A_n(\tau)e^{it} + \text{c.c.}$$

$$\Rightarrow \text{order } \varepsilon^{\alpha-1} : \quad \left[\partial_t^2 + 1\right]X^1 = -2i\partial_\tau A(\tau)e^{it} + \text{c.c.} + \delta^+ V'(\delta^- X^0)$$

$$X^1 \; 2\pi - \text{periodic in } t \Rightarrow \text{solvability condition } \int_0^{2\pi} e^{-it} \times \text{RHS } \, dt = 0$$

$$\Rightarrow \; 2i\,\partial_\tau A_n = f(A_{n+1} - A_n) - f(A_n - A_{n-1})$$

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-it}\, V'(ze^{it} + \bar{z}e^{-it})\,dt = \frac{1}{\tau_0}\, z\,|z|^{\alpha-1}$$

Newton's cradle vs discrete p-Schrödinger : error bounds

Infinite lattice ($n \in \mathbb{Z}$), phase space = sequence space ℓ_p with $p \in [1,\infty]$

The DpS equation approximates true O(ε) solutions of N.C. up to an error $O(\varepsilon^\alpha)$, over long times $O(\varepsilon^{1-\alpha})$:

Theorem (Bidégaray-Fesquet, Dumas, G.J. '13)

Fix a solution of DpS: $A_n(\tau) : [0,T] \to \ell_p(\mathbb{Z})$

For all ε small enough, the solutions of N.C. with initial conditions

$(x_n(0), \dot{x}_n(0))_n = (x_n^{A,\varepsilon}(0), \dot{x}_n^{A,\varepsilon}(0))_n + O(\varepsilon^\alpha)$ in $\ell_p^2(\mathbb{Z})$ satisfy :

$(x_n(t), \dot{x}_n(t))_n = (x_n^{A,\varepsilon}(t), \dot{x}_n^{A,\varepsilon}(t))_n + O(\varepsilon^\alpha)$ in $\ell_p^2(\mathbb{Z})$ uniformly in $t \in [0, T\varepsilon^{1-\alpha}]$

Method :

Consistency : $x^{A,\varepsilon} + O(\varepsilon^\alpha)$ correction solves N.C. up to an error $O(\varepsilon^{2\alpha-1})$,

Gronwall estimates for large times $t = O(\varepsilon^{1-\alpha})$

Error bound for DpS approximation : SKETCH OF PROOF

Known approximate solution : $x_{\text{app}}(t) = \left(\varepsilon X^0 + \varepsilon^\alpha X^1\right)(\varepsilon^{\alpha-1}t, t) = x^{A,\varepsilon}(t) + \mathrm{O}(\varepsilon^\alpha)$

Residual : $R(x_{\text{app}}) := \left(\frac{d^2}{dt^2} + 1 - \delta^+ V'(\delta^-)\right)(x_{\text{app}}) = \mathrm{O}(\varepsilon^{2\alpha-1}) + \boxed{\varepsilon^{3\alpha-2}\partial_\tau^2 X^1}$

No terms $\mathrm{O}(\varepsilon)$ and $\mathrm{O}(\varepsilon^\alpha)$ in $R(x_{\text{app}})$ with the choice of X^0 (solution of DpS) and X^1

Singular term $\partial_\tau^2 X^1$ (distribution !) must be eliminated:

N.C. equivalent to : $\frac{du}{dt} = J\,u + G(u)$, $u = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$, $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, $G = \begin{pmatrix} 0 \\ \delta^+ V'(\delta^-) \end{pmatrix}$

with $u(t) \in (\ell_p(\mathbb{Z}))^2$. Modified C^1 approximate solution $u_{\text{app}} := \begin{pmatrix} x_{\text{app}} \\ \dot{x}_{\text{app}} - \varepsilon^{2\alpha-1}\partial_\tau X^1 \end{pmatrix}$

Residual $E(u_{\text{app}}) := \left(\frac{d}{dt} - J - G\right)(u_{\text{app}}) = \begin{pmatrix} 0 \\ R(x_{\text{app}}) - \varepsilon^{3\alpha-2}\partial_\tau^2 X^1 \end{pmatrix} + \mathrm{O}(\varepsilon^{2\alpha-1}) = \mathrm{O}(\varepsilon^{2\alpha-1})$

Error $r := u - u_{\text{app}}$ satisfies $\frac{dr}{dt} = J\,r + G(u_{\text{app}} + r) - G(u_{\text{app}}) - E(u_{\text{app}})$. By Gronwall:

$$\|r(t)\| \le \|r(0)\| + \int_0^t \|E(u_{\text{app}})(s)\|\,ds + C\varepsilon^{\alpha-1}\int_0^t \|r(s)\|\,ds = \mathrm{O}(\varepsilon^\alpha) \text{ for } t = \mathrm{O}(\varepsilon^{1-\alpha})$$

□

Breather solutions of DpS (time-periodic) and long-lived breathers in Newton's cradle

$$\text{DpS}: \; i\partial_\tau A_n = (A_{n+1} - A_n)|A_{n+1} - A_n|^{\alpha-1} - (A_n - A_{n-1})|A_n - A_{n-1}|^{\alpha-1}$$

Time-periodic solutions to DpS : $A_n(\tau) = a_n e^{i\tau}$ $\boxed{a_n \in \mathbb{R}}$

Breathers : $\lim\limits_{n \to \pm\infty} a_n = 0$

Stationary (real) DpS equation :

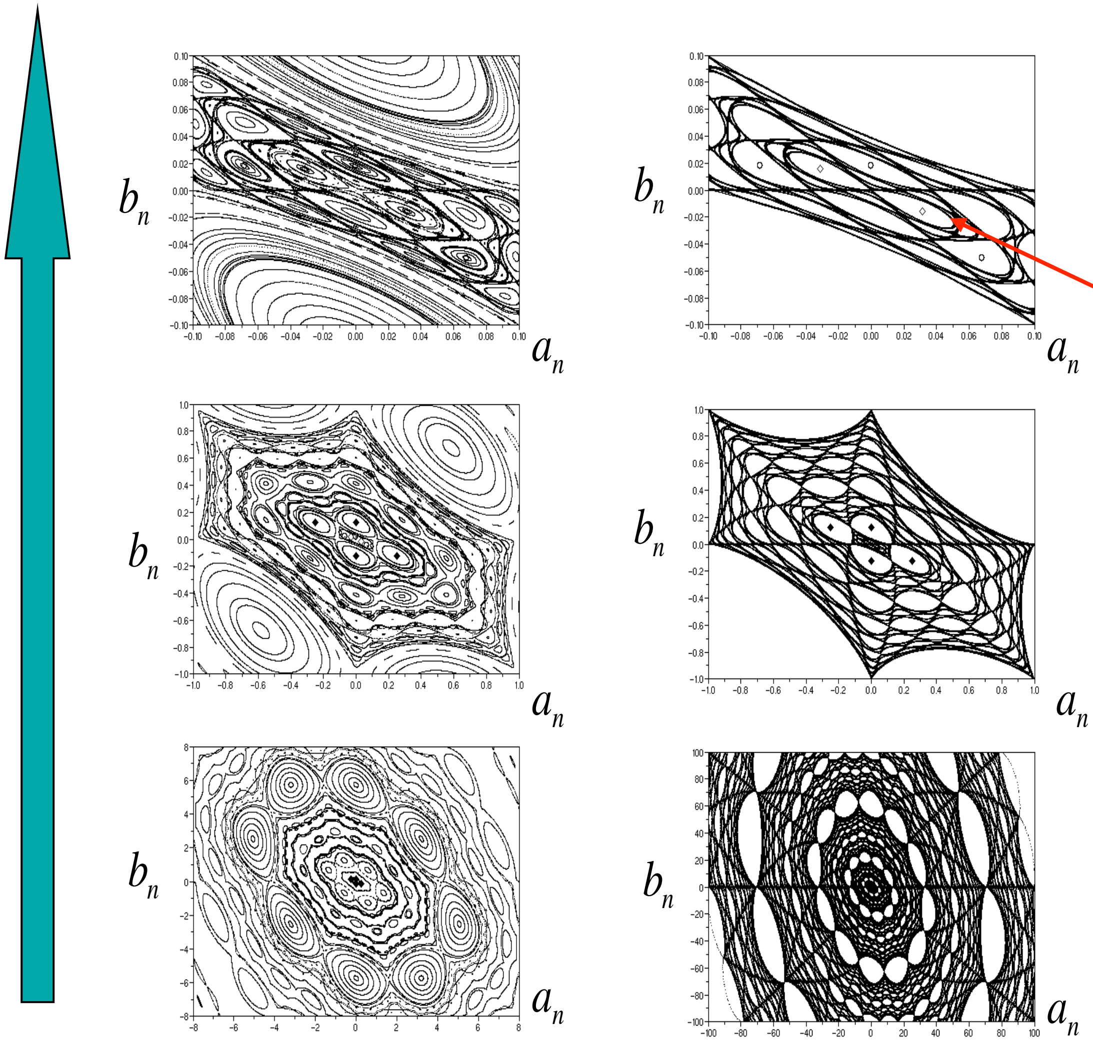
$$-a_n = \underbrace{(a_{n+1} - a_n)|a_{n+1} - a_n|^{\alpha-1}}_{b_n} - (a_n - a_{n-1})|a_n - a_{n-1}|^{\alpha-1}$$

Spatial dynamics : stationary real DpS ⇔ 2D mapping

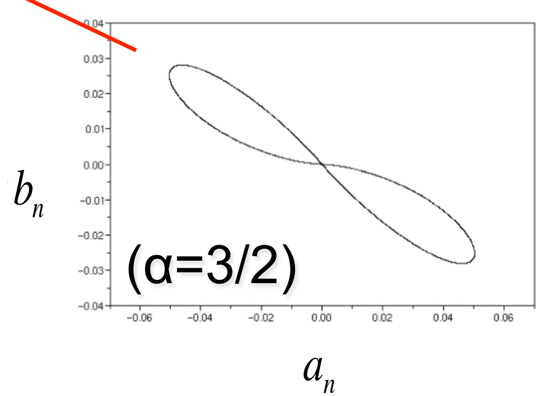
$(a_{n+1}, b_{n+1}) = G(a_n, b_n)$ G reversible, area-preserving, not differentiable at the origin

Stationary Dps equation : some orbits of the « spatial map » G

Zoom towards (0,0)



Stable and unstable manifolds of (0,0) intersect :

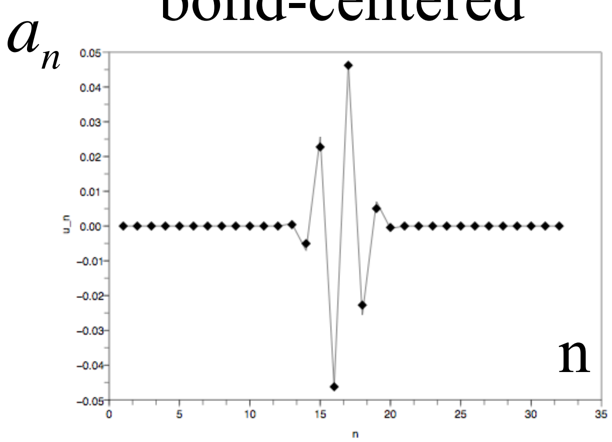


⇨orbits homoclinic to 0

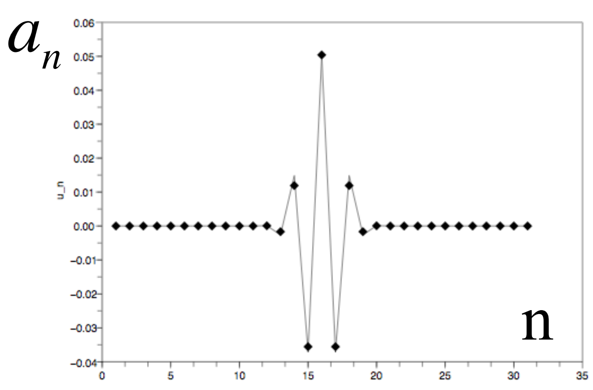
$$(a_n, b_n) \underset{n \to \pm\infty}{\longrightarrow} 0$$

Theorem : for all $\alpha > 1$, existence of reversible homoclinics $(a_n, b_n) \xrightarrow[n \to \pm\infty]{} 0$ (G.J. and Starosvetsky '14)

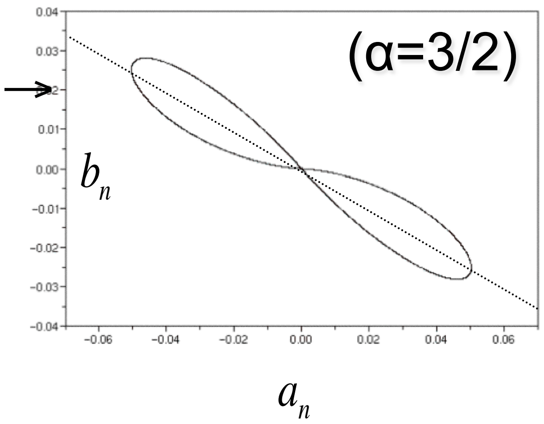
bond-centered



site-centered



Method : $W^{u}(0)$ and $W^{s}(0)$ intersect on a reversibility axis →



See also : Flach '95, Rosenau and Schochet '05, Qin and Xiao '07, Yoshimura '17 (periodic BC)

DpS breathers
⇨ long-lived breathers in Newton's cradle
(Bidégaray et al '13)

$$x_n(t) = 2\varepsilon\, a_n \cos\left[\left(1 + \frac{\varepsilon^{\alpha-1}}{2\tau_0}\right)t\right] + O(\varepsilon^{\alpha})$$

over long times $t \approx \varepsilon^{1-\alpha}$

Numerical computation of breathers in Newton's cradle

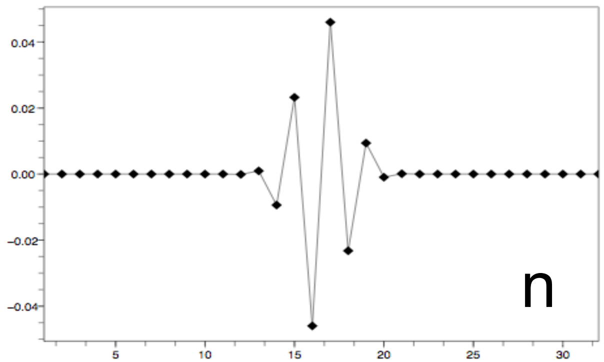
$$\ddot{x}_n + x_n = (x_{n-1} - x_n)_+^{3/2} - (x_n - x_{n+1})_+^{3/2}$$

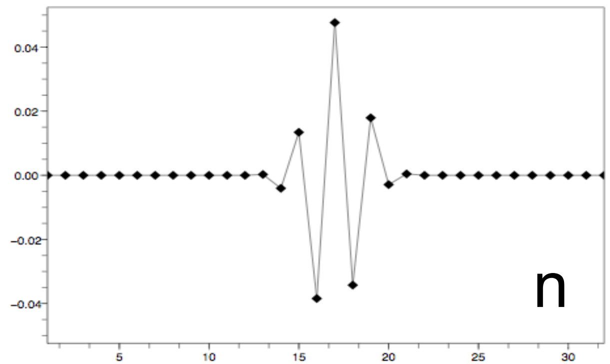
Breather solution : $x_n(t+T) = x_n(t), \quad \lim_{n\to\pm\infty} x_n(t) = 0$

Computation by Newton's method : (G.J., Kevrekidis, Cuevas '13)

Bond-centered breather   Site-centered breather

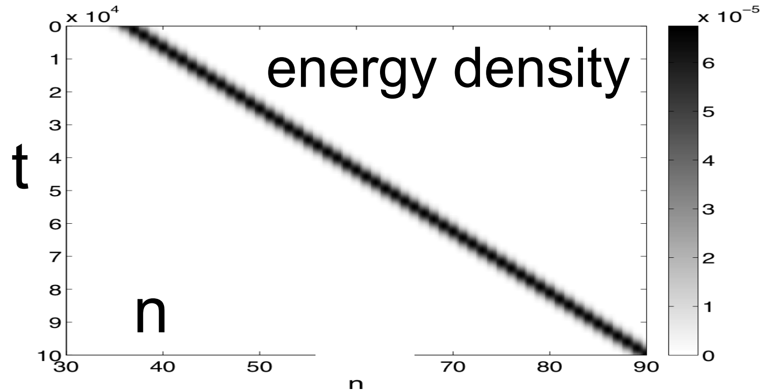
$x_n(0)$





$(T < 2\pi)$

Small perturbation (energy +0.01%) of a stable breather (bond-centered)
⇨traveling breather :



More on localized solutions of DpS : absence of complete scattering

DpS equation $(\alpha > 1)$:

$$i\,\partial_\tau A_n = (A_{n+1} - A_n)\,|\,A_{n+1} - A_n\,|^{\alpha-1} - (A_n - A_{n-1})\,|\,A_n - A_{n-1}\,|^{\alpha-1}$$

Theorem (Bidégaray-Fesquet, Dumas, G.J. ’13) :

DpS is globally well-posed in $\ell_2(\mathbb{Z})$. If $A(0) \neq 0$, then

for all times : $$\|A(\tau)\|_\infty \geq \left(\frac{\left\| \frac{1}{2}\delta^+ A(0) \right\|_{\alpha+1}^{\alpha+1}}{\|A(0)\|_2^2} \right)^{\frac{1}{\alpha-1}}$$

$$(\delta^+ A)_n = A_{n+1} - A_n, \qquad \|A\|_p = \left(\sum_{n=-\infty}^{+\infty} |\,A_n\,|^p \right)^{1/p}, \qquad \|A\|_\infty = \operatorname{Sup}_n |A_n|$$

Proof of the theorem

use two conserved quantities :

energy $H = \sum_{n=-\infty}^{+\infty} |A_{n+1} - A_n|^{\alpha+1}$ and $\|A\|_2^2 = \sum_{n=-\infty}^{+\infty} |A_n|^2$ (Kopidakis et al, '08)

$$\|\delta^+ A(0)\|_{\alpha+1} = H^{\frac{1}{\alpha+1}} = \|\delta^+ A(\tau)\|_{\alpha+1} \quad \text{(energy conservation)}$$

$$\leq 2\|A(\tau)\|_{\alpha+1} \quad \text{(triangular inequality)}$$

$$\leq 2\left(\|A(\tau)\|_\infty\right)^{1-\frac{2}{1+\alpha}} \left(\|A(\tau)\|_2\right)^{\frac{2}{1+\alpha}} \quad \text{(interpolation inequality)}$$

$$\leq 2\left(\|A(\tau)\|_\infty\right)^{1-\frac{2}{1+\alpha}} \left(\|A(0)\|_2\right)^{\frac{2}{1+\alpha}} \quad \text{(conserved } \ell_2 \text{ norm)}$$

$$\Rightarrow \|A(\tau)\|_\infty \geq \left(\frac{1}{2}\|\delta^+ A(0)\|_{\alpha+1}\right)^{\frac{\alpha+1}{\alpha-1}} \left(\|A(0)\|_2\right)^{\frac{2}{1-\alpha}} = \left(\frac{\left\|\frac{1}{2}\delta^+ A(0)\right\|_{\alpha+1}^{\alpha+1}}{\|A(0)\|_2^2}\right)^{\frac{1}{\alpha-1}} \qquad \square$$

III – DpS limit and breathers in MwM (heavy secondary mass)

$$\ddot{x}_n + x_n = (x_{n-1} - x_n)_+^{\alpha} - (x_n - x_{n+1})_+^{\alpha} + y_n$$

$$\ddot{y}_n = \gamma(x_n - y_n) \qquad \text{(MwM)}$$

$\alpha > 1$

$\gamma := 1/\rho = O(\varepsilon^{2(\alpha-1)})$

ε small parameter

Theorem (Liu et al '16)

phase space = $\ell_p^4(\mathbb{Z})$ with $p \in [1,\infty]$

Fix a solution of DpS: $A_n(\tau):\ [0,T] \to \ell_p(\mathbb{Z})$

For all ε small enough, the solutions of MwM with initial conditions

$(x_n(0), \dot{x}_n(0))_n = (x_n^{A,\varepsilon}(0), \dot{x}_n^{A,\varepsilon}(0))_n + O(\varepsilon^{\alpha}), \quad (y_n(0), \varepsilon^{1-\alpha}\dot{y}_n(0))_n = O(\varepsilon^{\alpha})$

satisfy uniformly in $t \in [0, T\varepsilon^{1-\alpha}]$:

$(x_n(t), \dot{x}_n(t))_n = (x_n^{A,\varepsilon}(t), \dot{x}_n^{A,\varepsilon}(t))_n + O(\varepsilon^{\alpha}), \quad (y_n(t), \varepsilon^{1-\alpha}\dot{y}_n(t))_n = O(\varepsilon^{\alpha})$

DpS breathers $\Rightarrow$ MwM breathers

over long times $t \approx \varepsilon^{1-\alpha}$:

$$x_n(t) = 2\varepsilon a_n \cos[(1 + \frac{\varepsilon^{\alpha-1}}{2\tau_0})t] + O(\varepsilon^{\alpha})$$

$$y_n(t) = O(\varepsilon^{\alpha}), \quad \dot{y}_n(t) = O(\varepsilon^{2\alpha-1})$$

Breakdown of DpS approximation over long times

No nontrivial T-periodic breather solutions satisfying $\lim_{n\to\pm\infty} \|x_n - x_{n-1}\|_{L^\infty(0,T)} = 0$

(Liu et al '16)

average interaction force : $f_n = \frac{1}{T}\int_0^T (x_{n-1} - x_n)_+^\alpha \, dt \to 0$ as $n \to \infty$ (localization)

MwM yields : $\ddot{x}_n + \rho\, \ddot{y}_n = (x_{n-1} - x_n)_+^\alpha - (x_n - x_{n+1})_+^\alpha$

$\Rightarrow \quad f_n$ independent of $n \quad \Rightarrow \quad f_n = 0$

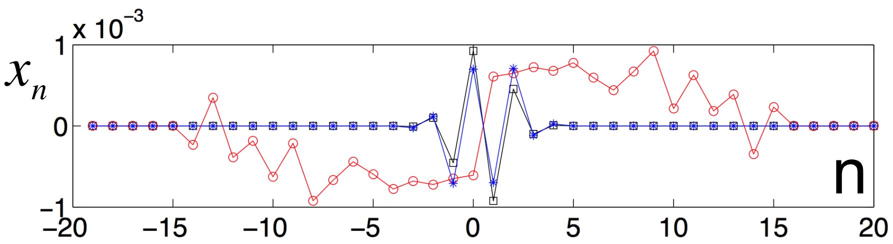
$\Rightarrow \quad (x_{n-1}(t) - x_n(t))_+^\alpha = 0 \;\; \forall t$ (beads not interacting) $\Rightarrow$ only trivial breathers

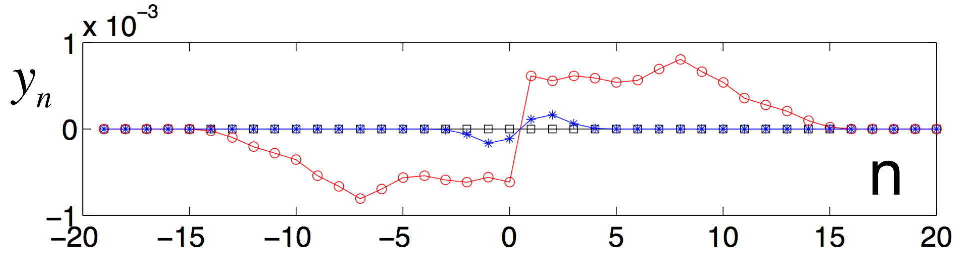
with $\ddot{x}_n = y_n - x_n, \;\; \rho\, \ddot{y}_n = x_n - y_n$ (freq. $\omega = \sqrt{1+\gamma}$) $\square$

Numerical simulation of MwM for $\rho = 1000$ ($\alpha = 3/2$)

Initial condition at $t = 0$: approximate DpS breather with $\varepsilon = 0.01$ (black)

Blue : evolution at $t = 36$ breather periods, red : $t = 80$ periods



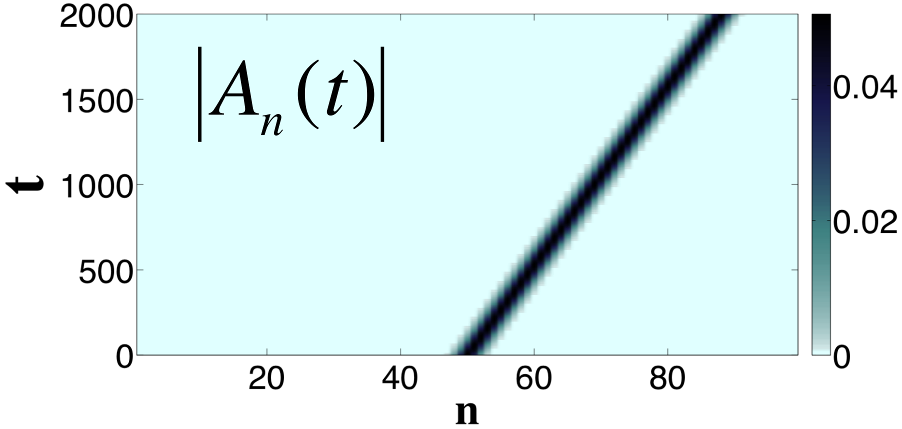


IV – Continuum limits of DpS and traveling breathers

$$i\frac{d}{dt}a_n = (\Delta_p a)_n, \quad n \in \mathbb{Z}, \qquad p = \alpha + 1 > 2$$

$$(\Delta_p a)_n = (a_{n+1} - a_n)\,|a_{n+1} - a_n|^{p-2} - (a_n - a_{n-1})\,|a_n - a_{n-1}|^{p-2}$$

Small perturbation of an unstable breather (site-centered) leading to translational motion, for p=5/2 :



Idea to describe traveling breathers :
use p-2 as small parameter for a « weakly nonlinear » analysis

See also : G.J. and Starosvetsky '14 (stationary breathers in DpS),

Chatterjee '99, G.J. and Pelinovsky '14 (solitary waves in granular chains)

Formal derivation of generalized NLS equations (G.J., '18)

$$i\frac{d}{dt}a_n = (\Delta_p a)_n, \quad n \in \mathbb{Z}, \qquad\qquad p \approx 2$$

scale invariance : $\quad a_n(t) \to R\, a_n(|R|^{p-2}\, t)$

Ansatz : $\quad a_n^{\mathrm{app}}(t) = \dfrac{R}{\sqrt{\Omega}}\, \phi_n(t)\, A(\xi,\tau)$

$$\phi_n(t) = e^{i\,(\Omega\,|R|^{p-2}\,t - q\,n)}$$

$$\Omega(q) = 4\sin^2(q/2), \quad q \in (0,\pi]$$

*Exact periodic traveling wave for A=1 (wavenumber q, amplitude R)

*Slow modulation in time and space for p ≈ 2

$$\xi = \sqrt{p-2}\,(n - c_q\,|R|^{p-2}\,t) \qquad\qquad c_q = \Omega'(q) = 2\sin q$$

$$\tau = (p-2)\,|R|^{p-2}\,t$$

Formal derivation of generalized NLS equations

Evaluation of residual error : $E := (i\frac{d}{dt} - \Delta_p)(a^{\mathrm{app}})$

$$= (p-2)\,\frac{R|R|^{p-2}}{\sqrt{\Omega}}\,\phi\,(i\partial_\tau A + \Omega\, N_p(A) - \cos q\,(\partial_\xi^2 A)|A|^{p-2} + \mathcal{O}(\sqrt{p-2})\,)$$

$$N_p(A) = A\,\frac{|A|^{p-2}-1}{p-2} = A\ln|A| + \mathrm{O}(p-2)$$

Amplitude equations yielding $E = \mathrm{O}((p-2)^{3/2})$:

logarithmic NLS equation : (Bialynicki-Birula and Mycielski ’76, Cazenave and Haraux ’80, …)

$$i\partial_\tau A = \cos q\,\partial_\xi^2 A - \Omega\, A\, \ln|A| \qquad \text{(log-NLS)}$$

fully-nonlinear Schrödinger equations :

$$i\partial_\tau A = \cos q\,(\partial_\xi^2 A)\,|A|^{p-2} - \Omega\, N_p(A) \qquad \text{(FNLS-I)}$$

$$i\partial_\tau A = \cos q\,\partial_\xi^2 (A\,|A|^{p-2}) - \Omega\, N_p(A) \qquad \text{(FNLS-II)}$$

NLS approximations of traveling breathers

Stationary equations FNLS-I and FNLS-II (Ahnert-Pikovsky '09) admit compactons for q∈(π/2,π] :

$$A_{\rm c}(\xi) = \begin{cases} A_1 |\cos(\lambda\xi)|^{\frac{2}{p-2}}, & |\xi| \le \frac{\pi}{2\lambda}, \\ 0, & |\xi| \ge \frac{\pi}{2\lambda}, \end{cases}$$

with constant coefficients $A_1(p),\ \lambda(p,q) = \mathrm{O}(\sqrt{p-2})$

➔ approximate traveling breathers for DpS with compact support :

$$a_n^{\rm app}(t) = \frac{R}{\sqrt{\Omega}}\, e^{i\,(\Omega\,|R|^{p-2}\, t - q\,n)}\, A_{\rm c}[\sqrt{p-2}\,(n - 2\,\sin q\,|R|^{p-2}\, t)]$$

log-NLS equation admits Gaussian solutions for q∈(π/2,π] :

$$A_{\rm g}(\xi) = \sqrt{e}\exp\left(\frac{\Omega}{4\,\cos q}\,\xi^2\right) = \lim_{p\to 2} A_{\rm c}(\xi)$$

Traveling breather solutions of the DpS equation : numerical computation

We compute localized initial conditions $a_n(0)$ in DpS such that :

$$\boxed{e^{i\theta}\, a_{n+1}(1/v) = a_n(0), \quad \forall n \in \mathbb{Z}} \quad \Rightarrow a_n(t) = a_0(t - n/v)\, e^{-i\theta n}$$

Parameters v, θ : traveling breather velocity and phase

Numerical tools : time-integration and Newton's method

Ansatz (compacton or Gaussian) :

$$a_n^{\mathrm{app}}(t) = \frac{R}{\sqrt{\Omega}}\, e^{i\,(\Omega\,|R|^{p-2}\,t - q\,n)}\, A_{\mathrm{c}}[\sqrt{p-2}\,(n - 2\,\sin q\,|R|^{p-2}\,t)]$$

Wavenumber q and amplitude R must satisfy :

$$q - \tan(q/2) = \theta\;(2\pi)$$

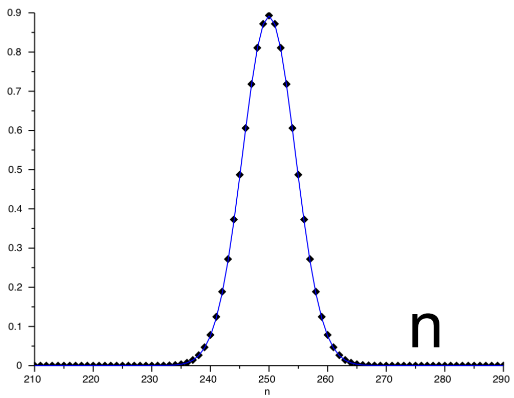
$$2\,\sin q\,|R|^{p-2} = v$$

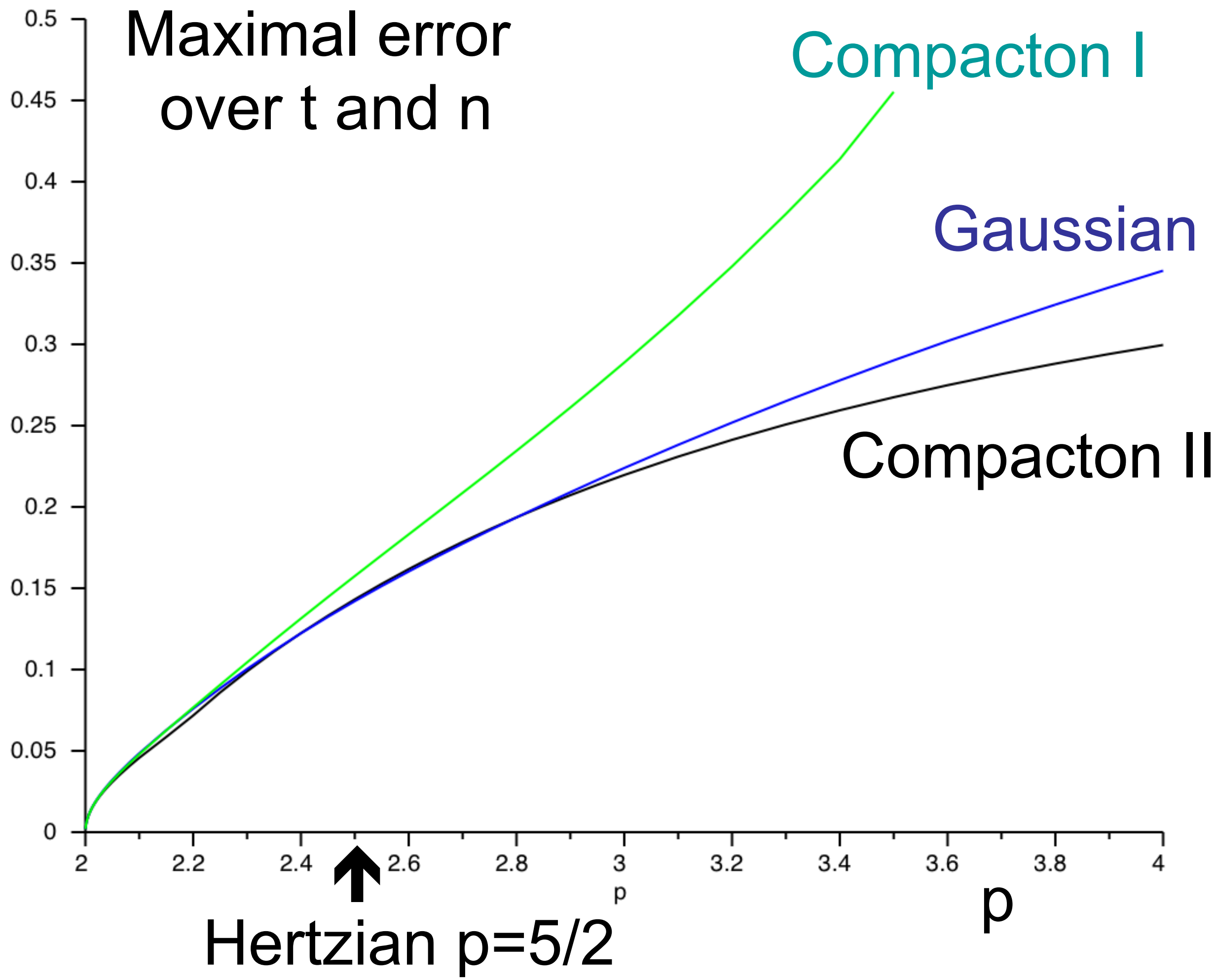
Numerical validation of NLS approximations

Relative error between Ansatz ($q = 3\pi / 4, R = 1$) and a numerically exact traveling breather $a_n(t) = a_0(t - n / v)\, e^{-i\theta n}$ (velocity $v = \sqrt{2}$, phase shift $\theta \approx -0.06$)

Numerical solution : Newton-type method
Aubry, Cretegny '98
Yoshimura, Doi '07

Gaussian profile of $|a_n(0)|$ for p = 2.02 :





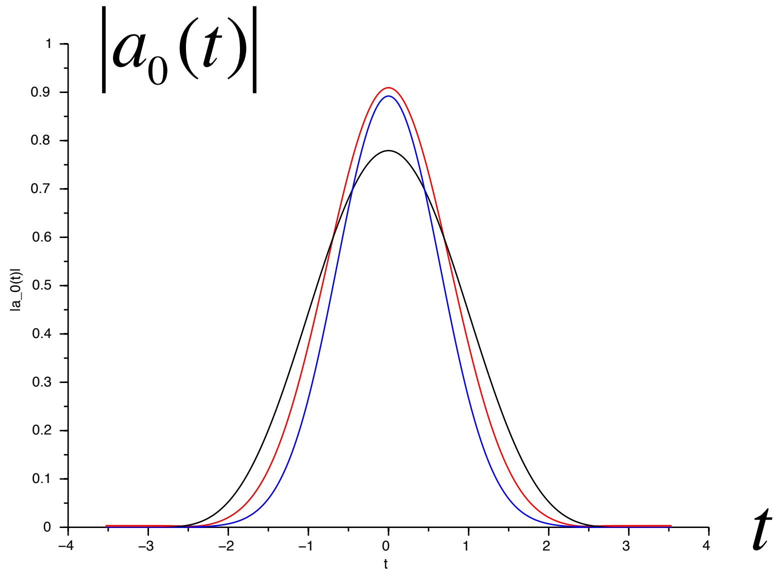
Numerical validation of NLS approximations

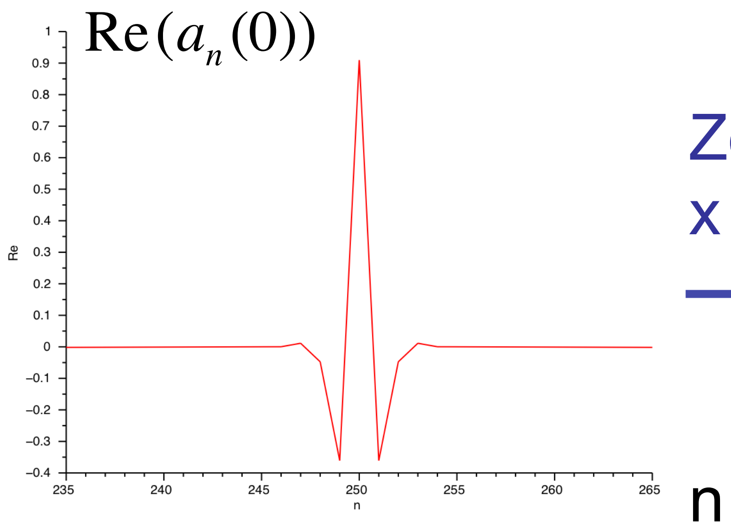
Example : p=5/2

numerical (Newton)

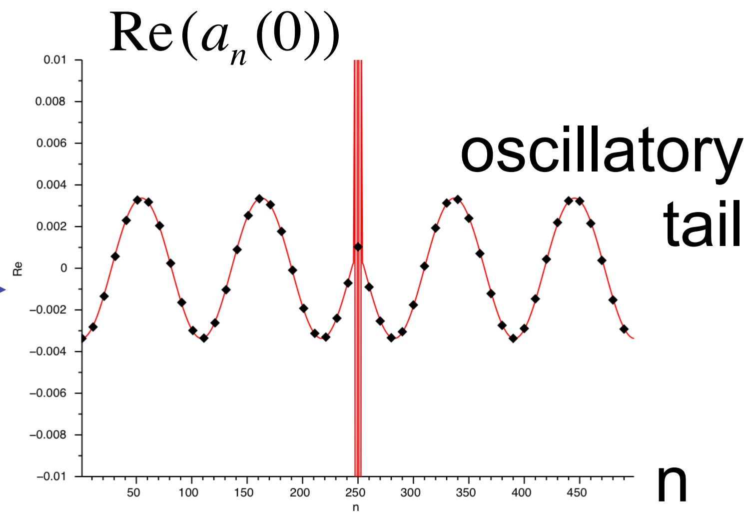
Gaussian

Compacton II



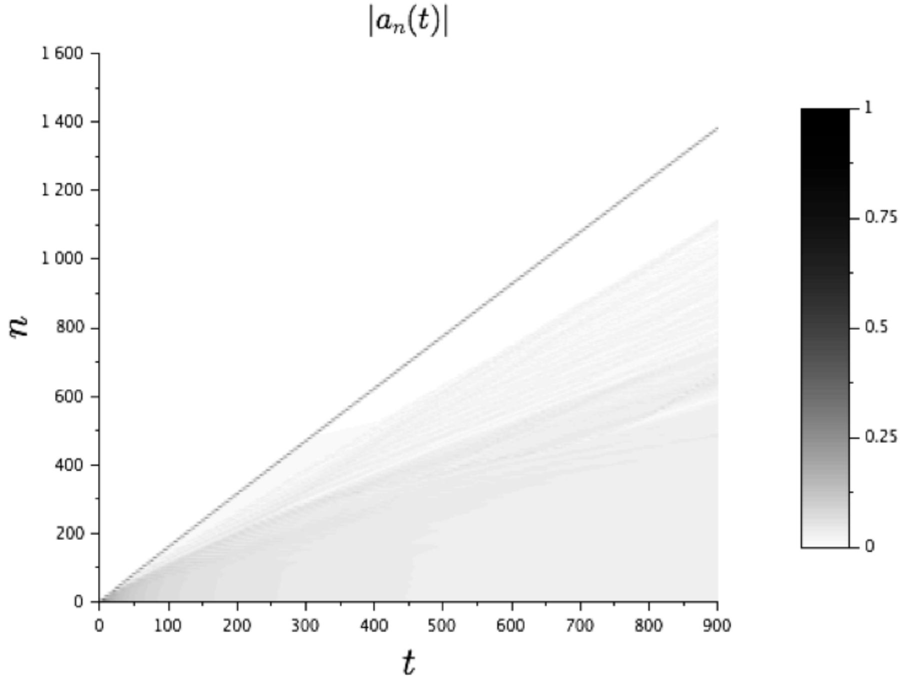


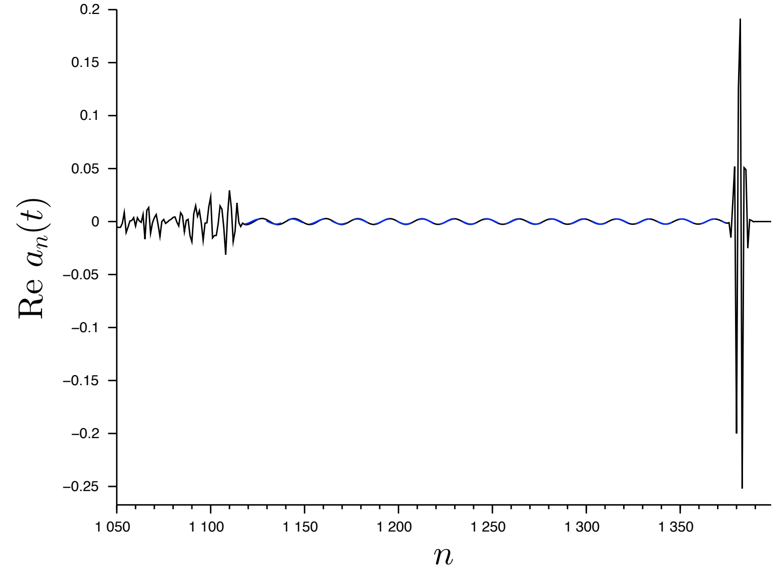
Zoom
x 100



Numerical validation of NLS approximations

Dynamical simulation for p=2.1 (initial perturbation of first particle)



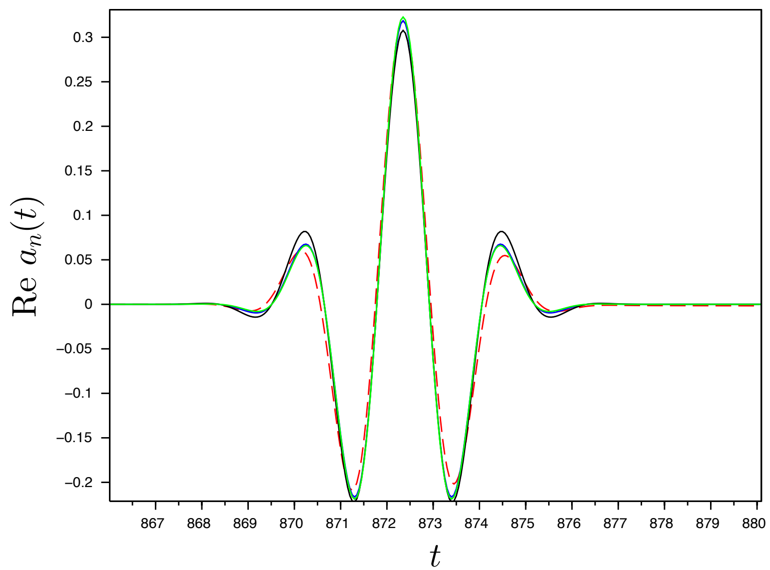


numerical time integr.

Gaussian

compacton I

compacton II



We fix q and R to match the numerical velocity $v \approx 1.5$ and phase $\theta \approx 0.3$

Conclusion

❖Nonlinear lattices modeling 1D granular metamaterials :

FPU (Hertz type potential, $p>2$) + local potential or attached masses

❖Time-periodic or transient (long-lived) breathers and traveling breathers obtained numerically

❖ asymptotic model : discrete p-Schrödinger equation (DpS)
-approximates small oscillations over long (but finite) times
-existence of time-periodic breathers
-traveling breathers approximated through formal continuum limits

Works in progress :

- error bounds in the multiscale analysis for $p \approx 2$
- dissipative impacts

References :

G.J., Math. Models Meth. Appl. Sci. 21 (2011), 2335-2377

G.J., P. Kevrekidis, J. Cuevas, Physica D 251 (2013), 39-59
+ proceedings of NOLTA 2012, arXiv:1301.1769

Bidégaray-Fesquet, Dumas, G.J., SIAM J. Math. Anal. 45 (2013), 3404-3430

G.J., Y. Starosvetsky, in « Localized Excitations in Nonlinear Complex Systems », Nonlinear Systems and Complexity 7, Springer, 2014

L. Liu, G.J., P. Kevrekidis, A. Vainchtein, Nonlinearity 29 (2016), 3496-3527.

G.J., Traveling breathers and solitary waves in strongly nonlinear lattices, to appear in Phil. Trans. Royal Soc. London A (2018).

Starosvetsky, Hasan, Vakakis, Manevitch, SIAM J. Appl. Math. 72 (2012), 337-361.

I. Daumont, T. Dauxois, M. Peyrard, Nonlinearity 10 (1997), 617-630.
L.I. Manevitch, Nonlinear Dynamics 25 (2001), 95-109.
Morgante, Johansson, Kopidakis, Aubry, Physica D 162 (2002), 53-94.
} Cubic DNLS