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THE MCKEAN'S CARICATURE OF THE FITZHUGH–NAGUMO MODEL I. THE SPACE-CLAMPED SYSTEM*

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Abstract. Within the context of Liénard equations, we present the FitzHugh–Nagumo model with an idealized nonlinearity. We give an analytical expression (i) for the transient regime corresponding to the emission of a finite number of action potentials (or spikes), and (ii) for the asymptotic regime corresponding to the existence of a limit cycle. We carry out a global analysis to study periodic solutions, the existence of which is linked to the solutions of a system of transcendental equations. The periodic solutions are obtained with the help of the harmonic balance method or as limit behavior of the transient regime. We show how the appearance of periodic solutions corresponds either to a fold limit cycle bifurcation or to a Hopf bifurcation at infinity. The results obtained are in agreement with local analysis methods, i.e., the Melnikov method and the averaging method. The generalization of the model leads us to formulate two conjectures concerning the number of limit cycles for the piecewise linear Liénard equations.

Key words. excitability, oscillations, limit cycle, piecewise linear model, bifurcation

AMS subject classifications. 34A05, 37G15, 34C05, 92C20

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1. Introduction.

We consider the autonomous system

$$\begin{aligned} \frac{dv}{dt} &= p(v) - w, \\ \frac{dw}{dt} &= bv, \end{aligned} \tag{1.1}$$

where $t \in \mathbb{R}$, $b > 0$, $v(t) \in \mathbb{R}$ represents the system status variable at time t, $w(t) \in \mathbb{R}$ represents an additional variable, and $p : \mathbb{R} \to \mathbb{R}$ is a given function. These equations are known as the Liénard system [23], [22]. Special cases of (1.1) provide mathematical models for many applications in science and engineering. We mention here biology [31], [17], electronics (e.g., the van der Pol model [38]), chemistry [19], and mechanics (for instance, damped mass spring systems).

In this paper we consider the case of a cubic-like function for p. System (1.1) then describes the behavior of an isolated excitable cell where v is the membrane potential and w the recovery variable. When p is given by

$$p(v) = v(1 - v)(v - a), \qquad \text{where} \quad 0 < a < 1, \tag{1.2}$$

system (1.1) is the polynomial FitzHugh–Nagumo model [8], [32]. It has given rise to many studies and the reader is referred to the references given in [31] and [17]. There are no particular requirements with respect to the choice of p, except to have a graphical representation similar to that given by (1.2). When p is a polynomial function, it is difficult to obtain analytical results since exact solutions cannot be obtained. In order to be able to go further with the study and the understanding of

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the model, we will follow the choice originally proposed by McKean [29], considering that

$$p(v) = -v + h(v-a), \qquad \text{where} \quad 0 < a < 1, \tag{1.3}$$

and h is the Heaviside function

$$h(x) = \begin{cases} 0 \text{ if } x < 0, \\ 1 \text{ if } x > 0. \end{cases} \tag{1.4}$$

The study of model (1.1)–(1.3) with a diffusive term on v was initiated by McKean [29] and developed considerably by Rinzel and Keller [34] and Wang [39], [40]. Their analyses covered the existence and the stability of traveling wave solutions.

The use of idealized nonlinearity with the help of the Heaviside function has become a classic procedure in the modeling of threshold effects in excitable media [3], [6], [16]. This approach leads to analytical results concerning properties of the model and provides a qualitative description for a more general class of functions. As far as we know, no specific studies have been carried out on the model isolated in space (1.1)–(1.3). More generally, we are going to study the following system:

$$\begin{aligned} \frac{dv}{dt} &= -\lambda v + \mu h(v-a) - w, \\ \frac{dw}{dt} &= bv, \end{aligned} \tag{1.5}$$

where

$$\lambda > 0,\ \mu > 0,\ a > 0,\ \text{and } \mu > \lambda a. \tag{1.6}$$

The latter condition shows the restriction that must be imposed upon p to obtain a shape similar to that obtained with (1.2). We are going to carry out a global analysis of equations (1.5) considering (λ, μ, a, b) as parameters. It should be noted that the change of variables

$$(\tilde{t}, \tilde{w}, \tilde{\lambda}, \tilde{\mu}) \to \frac{1}{\sqrt{b}}(t, w, \lambda, \mu) \tag{1.7}$$

enables us to consider the case $b = 1$. Nevertheless, we will not make this choice given the usefulness of the parameter b in the interpretation of the results. In addition, we are going to consider the case $b \to 0$.

Our study covers the case where a constant input I is injected into the system:

$$\begin{aligned} \frac{dv}{dt} &= p(v) - w + I, \\ \frac{dw}{dt} &= bv. \end{aligned}$$

We obtain (1.5) by putting $\tilde{w} = w - I$, which, in the phase plane, corresponds to a shift of the v-nullcline. The case of a variable current $I(t)$ will be discussed briefly and will be the subject of another paper. It should be noted that the FitzHugh–Nagumo model has an additional term in the recovery variable $\dot{w} = b(v - \gamma w)$, and the simplification $\gamma = 0$ introduces an artifact in the sense that a constant current does not change the behavior. However, since we are not interested in the bistable regime, this limiting situation allows a qualitative description of the excitable regime and captures the bifurcations of the complete system as $\gamma \to 0$.

This article is organized as follows. In section 2, we present the context into which we put our study and introduce the elements that are useful to our analysis. In section 3, we discuss the so-called *spike solution* that corresponds to the emission of a finite number of action potentials. Particular attention is given to the study of the singular perturbed system obtained as $b \to 0$. Section 4 is devoted to an analytical study of periodic solutions, and a geometric analysis is given in section 5. We determine, in section 6, an approximation of the bigger limit cycle. Section 7 provides a mathematical link between excitability and oscillations. In the final section, we summarize our results and we discuss the problem of the number of limit cycles for the piecewise linear Liénard equations.

2. General. First, let us consider system (1.1) with p having a cubic shape similar to that given by (1.2). For a smooth reaction function, $p \in C^1$, classical results from dynamical systems theory enable us to state the following proposition.

PROPOSITION 2.1. *The single fixed point $E_0 = (0, p(0))$ is locally stable if and only if $p'(0) < 0$. If $p'(0) \geq 0$, a limit cycle, surrounding E_0, appears via a Hopf bifurcation.*

Proof. The single fixed point of (1.1) is $(0, p(0))$. Its local stability is given by the eigenvalues of the Jacobian matrix of (1.1) at $(0, p(0))$:

$$J = \begin{bmatrix} p'(0) & -1 \\ b & 0 \end{bmatrix}.$$

For $p'(0) < 0$, we obtain local stability of the fixed point. The equality $p'(0) = 0$ corresponds to the Hopf bifurcation equation. The second part of the proposition is obtained by constructing an invariant set containing E_0 and using the Poincaré–Bendixson theorem. □

The Hopf bifurcation is a mechanism that is frequently encountered in the appearance of small-amplitude oscillations [13]. It is possible to specify the behavior of the solution in the neighborhood of its Hopf bifurcation and to obtain, locally, an analytical expression for the solution of system (1.1) [20]. However, the case that we are going to look at is the so-called *excitable* one, which corresponds to $p'(0) < 0$. There is no precise mathematical definition of excitability, and we say that a system is excitable if a perturbation from its resting state leads to a large excursion for the solution in the phase plane and a return to its resting state. This phenomenon is characterized by a solution $(v(t), w(t))$ of (1.5) satisfying the following two properties:

$$\begin{array}{ll} \text{(P1)} & \exists\, 0 < t_1 < t_2, \text{ so that } v \text{ is increasing on } [t_1, t_2], \\ \text{(P2)} & \lim_{t \to +\infty} v(t) = 0. \end{array}$$

Such a solution will be called a *spike solution*. It should be noted that (P2) is always satisfied when the domain of attraction of $(0, p(0))$ is the whole phase plane. For our study, property (P1) is sufficient to characterize the excitability of our system. When p is the function involved in (1.5), property (P1) will be satisfied as soon as (v, w) crosses the threshold segment $[-\lambda a, -\lambda a + \mu]$ in the phase plane. These two properties are in agreement with the characterization of excitability given in [1], i.e., the existence, in the phase space of the so-called *amplifying set* and *decaying set*.

We are going to examine the two phenomena associated with the emergence of an action potential. These phenomena will be written according to the concept of *spike* *solution* and periodic solution. The *spike solution* is a transient regime characterized by a finite number of action potentials. The periodic solution corresponds to the emission of an infinite number of action potentials. It is an asymptotic regime that shows the presence of a limit cycle. These two regimes represent the basic properties of neuronal excitability [33], [15].

Before proceeding with an analytical study of these regimes, we are going to give a qualitative interpretation of the dynamical behavior of system (1.1). This system can be rewritten in a convenient form usually used within the context of self-excited oscillations [30], [12]:

$$\frac{d^2v}{dt^2} - p'(v)\frac{dv}{dt} + bv = 0.$$

It is then useful to consider the energy derived from the harmonic oscillator (obtained as $p' = 0$) defined by

$$E = \frac{1}{2}\left(\frac{dv}{dt}\right)^2 + \frac{b}{2}v^2.$$

This gives a solution,

$$\frac{dE}{dt} = p'(v)\left(\frac{dv}{dt}\right)^2. \tag{2.1}$$

We find the stability of the fixed point $(0, p(0))$, provided by Proposition 2.1, when $p'(0) < 0$, which corresponds to damped oscillations in the neighborhood of this fixed point. As p' is not negative everywhere on $\mathbb{R}$, it is not possible to obtain a conclusion concerning the global stability of $(0, p(0))$. In particular, it is possible that the added energy, when $p' > 0$, is sufficient to give rise to a limit cycle. Equation (2.1) provides information concerning this cycle to the extent that it must contain at least one root of p'. Note that this result can be found using the Poincaré–Bendixson criterion. When p is the cubic polynomial proposed by FitzHugh–Nagumo (1.2), system (1.1) does not have a limit cycle [24]. Nevertheless, while keeping a similar shape for p, it is possible to obtain a limit cycle. For example, for

$$p(x) = \begin{cases} -x & \text{if } x \leq 0, \\ 10x(x-0.3)(1-x) & \text{if not,} \end{cases}$$

and $b = 6$, one observes, numerically, the existence of a stable limit cycle. Thus the constraint on p, said to be of cubic shape, leaves a variability in the dynamical behavior of (1.1). In the case we are going to study, where p is represented in Figure 2.1, we will see that the energy input due to the jump discontinuity of v', $\Delta v' = \mu$ when v crosses the line $v = a$, may be sufficient to give rise to a limit cycle. In this case, the limit cycle coexists with the fixed point $(0, 0)$ and this situation is termed as *hard self-excitation*.

Before beginning this study, it is necessary to specify the meaning given to a solution of (1.5) when p is discontinuous. Geometrically, a solution corresponds to a trajectory in the phase plane (v, w). If this trajectory crosses the line of discontinuity transversally, the solution is easy to define: for t such that $v(t) = a$, $v'(t)$ has a jump discontinuity ($v'(t^+) - v'(t^-) = \pm\mu$), and elsewhere the solution is C^1 and satisfies (1.5) in the classical sense. In the case where the trajectory tangentially meets the

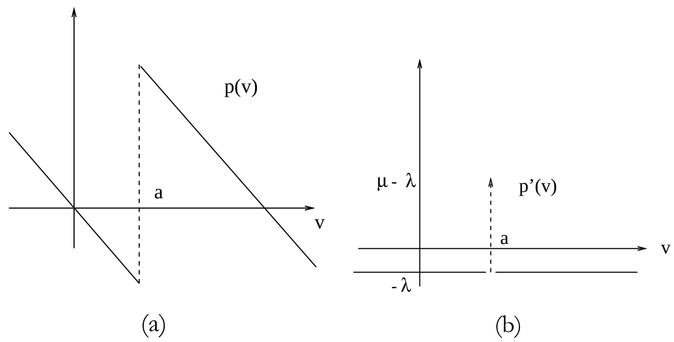

FIG. 2.1. (a) *The nonlinear function* p *and* (b) *its distributional derivative* p'.

line of discontinuity, the solution satisfies $v(t) = a$ on a nonempty set. In this case, we speak of a *generalized solution* and approach the problem from a geometrical point of view. It is not the purpose of this article to give a precise mathematical characterization of this solution, and the reader is referred to [7], [18]. It should be noted that the problem of discontinuous vector fields is covered extensively in control theory, e.g., [5], [14].

Our main results are given in the following summary. In section 3, we demonstrate that the spike solution contains only one spike when $\lambda^2 \geq 4b$ and several spikes can be emitted otherwise (depending on the initial conditions). In the former case, we derive a simple expression for the solution as $b \to 0$ and, in the latter, we give the general expression for the spike solution. The next sections focus on the case $\lambda^2 < 4b$ for which we derive, in section 4, analytical results on the existence and the expression of the periodic solutions. The periodic orbits appear via a double limit cycle bifurcation that we compute in the plane (a, b). Using a geometrical analysis (section 5), we characterize the two periodic solutions, represented in the phase plane by two concentric limit cycles. We show how in the limiting situation $\lambda \to 0$ and $\mu \to 0$ the periodic orbits can be obtained with the use of the Melnikov function. Moreover, we discuss the existence of two different types of unstable limit cycles referred to either as a classical or a generalized solution. The generalized solution is related to the discontinuity of the vector field. In section 6, the study as $\lambda \to 0$ allows us to capture and to describe the bigger limit cycle which is obtained as a Hopf bifurcation at infinity. In section 7, we show how the spike solutions and the periodic solutions are related.

3. Excitability and singular perturbation. The purpose of this section is to study the *spike solution*. In particular, we characterize this solution by the number of spikes that are part of the solution. This number corresponds to the number of times that v crosses the threshold a, where $v'^{-} > 0$ (where v'^{-} designates the left-hand derivative of v). This number includes the initial pulse corresponding to the perturbation due to the initial condition, noted as (v_0, w_0). In order to simplify the study we consider the case where $w_0 = 0$. We distinguish between several cases, according to the value of $\lambda^2 - 4b$. We prove the following proposition.

PROPOSITION 3.1. *For* $\lambda^2 \geq 4b$, *there is a spike solution when* $a < v_0 < \frac{\mu}{\lambda}$. *This solution only presents a single spike.*

Proof. First, we look at the case where $\lambda^2 > 4b$. For $v_0 < a$ and as long as $v(t) < a$, we have

$$v''(t) + \lambda v'(t) + bv(t) = 0.$$

The solution is then given by

$$v(t) = \frac{v_0}{r_+ - r_-}\left(-(\lambda - r_-)e^{r_+ t} + (\lambda + r_+)e^{r_- t}\right),$$

where

$$r_\pm = \frac{1}{2}(-\lambda \pm \sqrt{\lambda^2 - 4b}). \tag{3.1}$$

Let $\tilde{t}$ be the time defined by $\tilde{t} = 1/(r_+ - r_-)\ln(1 + \frac{\lambda}{r_+})/(1 + \frac{\lambda}{r_-})$. If $v_0 > 0$, v is decreasing on $[0, \tilde{t}\,]$ and increasing on $[\tilde{t}, +\infty[$. In addition, we have $\lim_{t\to+\infty} v(t) = 0$ and thus $\forall t > 0, v(t) < a$. When $v_0 < 0$, if $\forall t, v(t) < a$, the study is completed; conversely, if there is a time t^* so that $v(t^*) = a$, then the trajectory crosses the line $w = 0$ for a value of v greater than a, and, given a time shift, the study corresponds to the case where $v_0 > a$.

If $v_0 > a$, there is a time t^* so that $v(t^*) = 0$ and $w(t^*) = w_1 > -\lambda a + \mu$. If we put $t^* = 0$, this gives $v(t) = \frac{w_1}{r_+ - r_-}(e^{r_- t} - e^{r_+ t})$ and thus $\forall t > 0$, $v(t) < 0$ and $\lim_{t\to+\infty} v(t) = 0$.

The case where $\lambda^2 = 4b$ is dealt with in a similar way. □

For $\lambda^2 - 4b \geq 0$, the response to an input $I = I_0\delta(t - t_0)$ is a single action potential when $I_0 > a$.

We will now obtain a simple analytical expression for the potential v. The previous study showed the existence of several phases when a spike is emitted. This point can be made more specific by studying the case $b \ll 1$. This situation models the behavior of a system in which two time scales are involved; i.e., v is a fast variable and w is a slow variable. The mathematical description of the excitability is a classical one (see, for example, [17]) and is carried out using the singular perturbation theory. In our case, the relevance is to allow explicit solutions that give a simple expression for v according to the different phases of the *spike solution*.

Let there be (v_0, w_0) so that $a < v_0 < \frac{\mu}{\lambda}$ and $w_0 = 0$. In addition, let us assume that $v_0 - \frac{\mu}{\lambda}$ is of order greater than a $O(b)$. The variations of v can be separated into four phases. The first phase, which is the *excited phase*, is fast and the motion is governed approximatively by the system

$$\frac{dv}{dt} = p(v) - w,$$
$$\frac{dw}{dt} = 0,$$

which gives

$$v(t) = \frac{1}{\lambda}\left(\mu + e^{-\lambda t}(\lambda v_0 - \mu)\right). \tag{3.2}$$

This approximation is valid as long as $v(t)$ is at a greater distance from the v-nullcline than a $O(b)$ value. If not, we enter the second phase where the dynamic is described using a new time scale $\tau = bt$. In this phase, v is adjusted to maintain a pseudoequilibrium at $w = p(v)$, and we have $v(\tau) = \frac{1}{\lambda}(\mu - w)$. We obtain

$$v(t) = \frac{\mu}{\lambda}e^{-\frac{\tau}{\lambda}}. \tag{3.3}$$

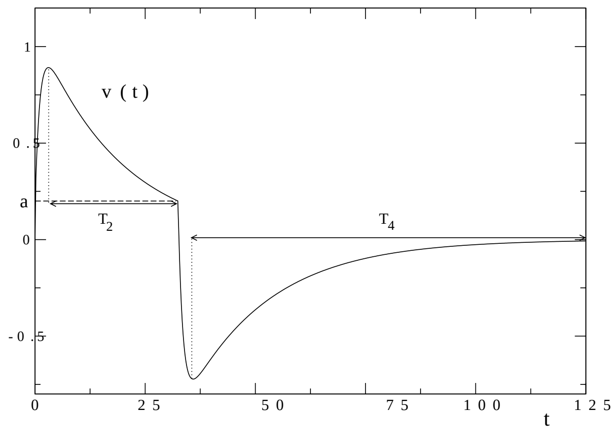

FIG. 3.1. *Solution $v(t)$ of (1.5) for $(v_0, w_0) = (0.25, 0)$, $\lambda = 1$, $a = 0.2$, $b = 0.05$, and $\mu = 1$. Intervals T_2 and T_4 designate the durations of the two slow phases.*

We enter into the third phase as v reaches a. We have a fast motion where v is given by

$$v(t) = \frac{\mu}{\lambda}(e^{-\lambda t} - 1) + a. \tag{3.4}$$

The final phase is characterized by a slow return to the equilibrium state according to

$$v(t) = \left(a - \frac{\mu}{\lambda}\right) e^{-\frac{\tau}{\lambda}}. \tag{3.5}$$

We can easily find these results by observing that the roots r_+ and r_- given by (3.1) are written $r_+ = -\frac{b}{\lambda} + O(b^2)$ and $r_- = -\lambda + O(b)$. The fast dynamic is obtained using the zero order approximation, and the slow motion by using the first order one. The different phases, (3.2)–(3.5), correspond to the charge and discharge of a capacitor, and are graphically shown in Figure 3.1. They allow precise identification of the role of each parameter. In particular, the amplitude of the potential is parameterized by $\frac{\mu}{\lambda}$ and a. In addition, it is possible to obtain an approximation of the duration of a spike T using the durations of the slow dynamics of phases two and four, written T_2 and T_4, respectively. We consider that the duration of phase four is the time for which $v(t) = O(b)$. This gives

$$T = T_2 + T_4,$$

where

$$\begin{aligned} T_2 &= \frac{\lambda}{b} \ln \frac{\mu}{\lambda a} + O\left(\frac{1}{b}\right), \\ T_4 &= O\left(-\frac{\ln b}{b}\right). \end{aligned} \tag{3.6}$$

For $b \ll 1$, it is possible to obtain a simple description of the subthreshold response to a variable input $I(t)$. This response is the one given by an RC filter, where $\lambda = \frac{1}{RC}$ and is written

$$v(t) = e^{-\lambda \cdot} * I(t).$$

If we consider a train of impulses at regular intervals, the system reacts preferentially at a high input frequency in that the higher the input frequency, the earlier a spike is emitted. More precisely, with $I(t) = I_0 \sum_{t_i} \delta(t - t_i)$, where $t_i = iT$ and $i \in \mathbb{N}$, the subthreshold response is given by

$$v(t) = I_0 e^{-\lambda t} \frac{1 - e^{\lambda(n+1)T}}{1 - e^{\lambda T}},$$

where n is the index of the final pulse of $I(t)$ before the system reaches the threshold. Thus, for an input such as $I_0 < a$ with a small frequency

$$\frac{1}{T} < \frac{\lambda}{\ln(\frac{a}{a - I_0})},$$

the system cannot emit an action potential.

Now, and for the rest of this article, unless indicated otherwise, we are going to consider the case in which $\lambda^2 - 4b < 0$. We shall see that the model presents a richer dynamic in the sense that the *spike solution* is able to present several action potentials. In addition, we will show in the following section the existence of periodic solutions. We continue the case of a solution satisfying $\lim_{t \to +\infty} v(t) = 0$ and, therefore, there exists a constant $C > 0$ and a time t^* starting from which we have

$$|v(t)| < C e^{-\frac{\lambda}{2} t}.$$

We can then define the Laplace transform of v

$$\mathcal{L}(v)(p) = \int_0^\infty v(t) e^{-pt} dt$$

for which the region of convergence is the half plane

$$D = \left\{ p \in \mathbb{C} \mid \mathrm{Re}(p) > -\frac{\lambda}{2} \right\}.$$

We define the finite sequence of times, written $(t_i)_{0 \dots 2n-1}$, so that $t_0 = 0$, and for $i \neq 0$, $v(t_i) = a$ and $\Delta v'(t_i) = (-1)^i \mu$. This sequence indicates the passage of potential via the line of discontinuity and corresponds to a jump of the derivative of v. An equivalent characterization of t_i is given by $v'(t_{2j}^-) > 0$ and $v'(t_{2j+1}^-) < 0$. We have, on $]t_{2i}, t_{2i+1}[$, $v(t) > a$ with $v(t_0) = v_0 > a$. The number n corresponds to the number of spikes emitted by the system. For $w_0 = 0$, we calculate

$$\mathcal{L}(v)(p) = \frac{\mu}{p^2 + \lambda p + b} \sum_{i=0}^{n-1} (e^{-pt_{2i}} - e^{-pt_{2i+1}}) + \frac{p v_0}{p^2 + \lambda p + b}.$$

We write in the following $r = \sqrt{4b - \lambda^2}$. Using inverse Laplace transforms gives

$$v(t) = v_0 \alpha(t) + \sum_{i=0}^{2n-1} (-1)^i h(t - t_i) \varphi(t - t_i), \tag{3.7}$$

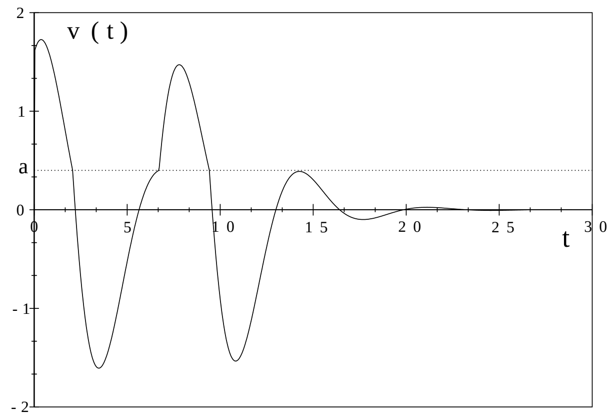

FIG. 3.2. *Solution* $v(t)$ *of* (1.5) *for* $(v_0, w_0) = (1.6, 0)$*,* $\lambda = 0.8$*,* $\mu = 2$*,* $a = 0.4$*, and* $b = 1$*. This* spike solution *presents two action potentials, or spikes.*

where

$$\varphi(t) = \frac{2\mu}{r} e^{-\frac{\lambda}{2}t} \sin \frac{r}{2}t, \tag{3.8}$$

$$\alpha(t) = e^{-\frac{\lambda}{2}t}\Big(\cos\frac{r}{2}t - \frac{\lambda}{r}\sin\frac{r}{2}t\Big). \tag{3.9}$$

The above expression characterizes the transient regime with (i) a term that depends on the initial excitation, v_0, and corresponds to damped oscillations of period $\frac{4\pi}{\sqrt{4b-\lambda^2}}$, and (ii) a sum of terms with the form $(-1)^i S_{t_i}(h\varphi)$ (where S is the shift operator) reproducing an excitation when $v'^- > 0$ (even i) or an inhibition $v'^- < 0$ (odd i). This sum shows the different crossing $v = a$ and is defined implicitly by the existence of times t_i such as $v(t_i) = a$. It is clear that t_1 exists: it is given by the smallest strictly positive solution of the equation

$$v_0\alpha(t) + \varphi(t) = a.$$

The sequence of times (t_i) cannot be expressed with known functions and is implicitly defined using expression (3.7). Figure 3.2 illustrates the case in which the system generates two action potentials. The return to the resting state takes place via damped oscillations and induces computational properties which differ from that studied above. If we consider the case of a system that has not emitted a spike, its subthreshold response to an input $I(t)$ is given by

$$v(t) = \alpha * I,$$

corresponding to the response of an RLC filter, with $\lambda = \frac{R}{L}$ and $b = \frac{1}{LC}$, when an input I is applied. In particular, the filter response is more significant for an input signal having a resonant frequency close to $\sqrt{b - \frac{\lambda^2}{4}}$.

4. Periodic solutions. Let us assume that system (1.5) has a periodic solution. According to the expression of the vector field, this solution delimits a domain containing the origin, which is a stable fixed point. In addition, when $\lambda \neq 0$, it is possible to construct an invariant region large enough to include this limit cycle. Thus, there are at least two limit cycles surrounding the origin, with an alternation of stable and unstable cycles, the largest being stable.

We are looking for a periodic solution $(v(t), w(t)) \in \left(L^2(0,T)\right)^2$, where T is the period of the solution. This solution can be expressed in a Fourier series

$$\begin{aligned} v(t) &= \sum_n v_n e^{2i\pi n \frac{t}{T}}, \\ w(t) &= \sum_n w_n e^{2i\pi n \frac{t}{T}}. \end{aligned} \tag{4.1}$$

The technique used, which is known as the method of the harmonic balance (see [2], for example), involves identifying (v_n, w_n) using the differential equation satisfied by (v, w).

In the phase plane, a periodic solution crosses the line $v = a$ at two points, one of which satisfies $w > 0$ and the other $w < 0$. We set t_1 and t_2 as the two successive times that satisfy $v(t_i) = a,\ i = 1, 2$, so that $\forall t \in]t_1, t_2],\ v(t) > a$. The time-translation invariance of the periodic solution allows us to define the real τ so that $t_1 = -\tau$, $t_2 = \tau$, where $0 < \tau < \frac{T}{2}$. The periodic solution looked for satisfies

$$v(t) = \begin{cases} > a & \text{on }]-\tau, \tau[, \\ < a & \text{on } [-\frac{T}{2}, -\tau[\cup]\tau, \frac{T}{2}]. \end{cases} \tag{4.2}$$

The function $t \to h(v(t) - a)$ is a T-periodic function such as

$$h(v(t) - a) = \begin{cases} 1 & \text{if } t \in [-\tau, \tau], \\ 0 & \text{if not,} \end{cases}$$

and we calculate that

$$h(v(t) - a) = \frac{2\tau}{T} + \sum_{n \neq 0} \frac{1}{\pi n} \sin\left(2\pi n \frac{\tau}{T}\right) e^{2i\pi n \frac{t}{T}}.$$

Therefore we obtain

$$v(t) = \sum_n c_n \sin\left(2\pi n \frac{\tau}{T}\right) e^{2i\pi n \frac{t}{T}}, \tag{4.3}$$

where

$$c_n = \frac{2\mu T i}{-4\pi^2 n^2 + bT^2 + i2\pi\lambda T n}.$$

At this stage in the study, we may remark that the mean value of v is zero (which could be seen directly with (1.5)). The mean value of w is $w_0 = \frac{2\mu\tau}{T}$. The amplitude spectrum of v is $O\left(\frac{1}{n^2}\right)$, which ensures the normal convergence of the associated Fourier series.

Let f be the function defined by

$$f(t) = \sum_n i c_n e^{i2\pi n \frac{t}{T}}.$$

With the help of trigonometric transformations, (4.3) is written

$$v(t) = \frac{1}{2}(f(t - \tau) - f(t + \tau)). \tag{4.4}$$

We calculate

$$f(t) = \frac{-2\mu}{r\left(\cosh\frac{\lambda T}{2} - \cos\frac{r}{2}T\right)} e^{\frac{\lambda}{4}(T-2t)} \left(e^{-\frac{\lambda T}{4}} \sin\frac{r}{2}(T-t) + e^{\frac{\lambda T}{4}} \sin\frac{r}{2}t\right) \tag{4.5}$$

for $0 \le t \le T$, where $r = \sqrt{4b - \lambda^2}$ and f is defined on $\mathbb{R}$ by periodicity. Therefore, f is continuous on $\mathbb{R}$ and has a derivative for $t \neq \mathbb{Z}T$. Note that f does not depend on the auxiliary variable τ. A periodic solution exists if and only if there is T and τ such as $0 < \tau < \frac{T}{2}$, solutions of

$$\begin{aligned} f(0) - f(2\tau) &= 2a, \\ f(-2\tau) - f(0) &= 2a, \end{aligned} \tag{4.6}$$

so that v, given by (4.4), satisfies (4.2). We note $x = \frac{T}{2}$ and $y = \tau$. Elementary operations show that (4.6) can be written in the form

$$\begin{aligned} F(x, y) &= 0, \\ F(x, y - x) &= 0, \\ 0 < &y < x, \end{aligned} \tag{4.7}$$

where

$$F(x, y) = \mu \sinh \lambda x \sin ry - \mu \sin rx \sinh \lambda y - ar(\cosh \lambda x - \cos rx) \cosh \lambda y. \tag{4.8}$$

The existence of periodic solutions for the differential system (1.5) is given by the existence of roots for a system of transcendental equations. We have therefore reduced the differential problem to an algebraic one that corresponds to a search for roots in $\mathbb{R}$. In contrast to perturbation methods, it is interesting to note that our analysis is a global one and gives an analytical formula for a periodic solution.

Remark. A similar study can be carried out for $\lambda^2 - 4b > 0$. We then write $r = \sqrt{\lambda^2 - 4b}$. We find that

$$f(t) = \frac{-2\mu}{r\left(\cosh\frac{\lambda T}{2} - \cosh\frac{r}{2}T\right)} e^{\frac{\lambda}{4}(T-2t)} \left(e^{-\frac{\lambda T}{4}} \sinh\frac{r}{2}(T-t) + e^{\frac{\lambda T}{4}} \sinh\frac{r}{2}t\right),$$

where T and τ are given by the resolution of (4.7) with F defined by

$$F(x, y) = \mu \sinh \lambda x \sinh ry - \mu \sinh rx \sinh \lambda y - ar(\cosh \lambda x - \cosh rx) \cosh \lambda y. \tag{4.9}$$

Using $r < \lambda$, it is easy to show that $F(x, y - x) < 0$ when $0 < y < x$, and, therefore, there cannot be solutions of (4.7) with F given by (4.9), which confirms the result of the previous section.

Starting from the study carried out above, it is possible to state several simple properties concerning a periodic solution. First of all, it is easy to see that its existence is controlled by parameters r, λ, and $\frac{a}{\mu}$. In addition, we have the following bound for the periodic solution:

$$\|v\|_{+\infty} < \frac{4\mu}{\sqrt{4b - \lambda^2}} \cdot \frac{e^{\frac{\lambda T}{2}} + 1}{e^{\frac{\lambda T}{2}} - 2},$$

which is valid for $e^{\frac{\lambda T}{2}} > 2$. In particular, we can see that the smaller the period, the larger the bound.

Based on (4.7), in the general case, it is difficult to give conditions for the existence of (T, τ). More precisely, two phenomena appear to make the study tricky: (i) the presence of solutions of (4.7) that do not correspond to a periodic solution, and (ii) the presence of a periodic solution not detected by our analysis. We will look more closely at the second point in the next section. The first point arises from the fact that the existence of exactly two solutions for the equation $v(t) = a$ on $[0, T]$, corresponding to (4.2), is not reported in system (4.7). These two situations can be illustrated by looking at the solutions of (4.7) as $a \to 0$. If we take $a = 0$, the resolution of (4.7) leads to the family of solutions $(T_k, \tau_{p_k})_k$, where $k \in \mathbb{N},\ k > 1$:

$$\begin{aligned} T_k &= \frac{2k\pi}{\sqrt{4b - \lambda^2}}, \\ \tau_{p_k} &= \frac{p_k \pi}{\sqrt{4b - \lambda^2}}, \quad \text{where } p_k = 1 \ldots k-1, \end{aligned} \tag{4.10}$$

and the implicit functions theorem leads to the existence of these solutions for a sufficiently small a. In fact, only the solution obtained for $k = 2$ is admissible; other solutions do not satisfy the assumptions of our study (given by (4.2)). Numerically, this solution corresponds to a stable limit cycle. As we have already mentioned, there must be an unstable cycle separating the domain of attraction of the origin from the stable cycle one. Therefore, we are in a situation where a limit cycle has not been detected.

Before clarifying this situation, we carry out a numerical study of the specific case used in [29], [34], and [39], where $\mu = 1$ and $\lambda = 1$. The results are illustrated in Figure 4.1, where we determine in the plane (a, b) the region where a periodic solution exists. It appears that there is a value of a, noted a^*, for which there is no periodic solution for $a \geq a^*$. When $a < a^*$, the existence of a periodic solution is obtained for $b \geq b_f(a)$. The curve $b_f(a)$ is given by the resolution, in $\{(x, y) \in \mathbb{R}^2 /\ 0 < y < x\}$, of

$$\begin{aligned} F(x, y) &= 0, \\ F(x, x - y) &= 0, \\ \det\, F_{x,y} &= 0, \end{aligned} \tag{4.11}$$

where $F_{x,y}$ is the Jacobian matrix of the system above with respect to (x, y). Geometrically speaking, the latter condition corresponds to a tangential intersection between the two curves defined by the equations $F(x, y) = 0$ and $F(x, x - y) = 0$. For $b = b_f(a)$, there is a single unstable limit cycle. For $b > b_f(a)$, there are two concentric limit cycles. The larger one is stable, and the smaller one is unstable, separating the different domains of attraction. At $b = b_f(a)$ a fold limit cycle bifurcation (or double limit cycle bifurcation) occurs. Several limiting situations can be analytically specified. When $a \to 0$, system (4.7) always has an admissible solution (given by (4.10) with $k = 2$), and the only restriction on b is related to the existence of r. We therefore have $\lim_{a \to 0} b_f(a) = 0.25$.

We determine the value of a^* using an asymptotic expansion of (4.7) as $b \to +\infty$. More exactly, we use an asymptotic expansion as $r \to +\infty$.

We write

$$x = \frac{x_1}{r} + O\left(\frac{1}{r^2}\right)$$

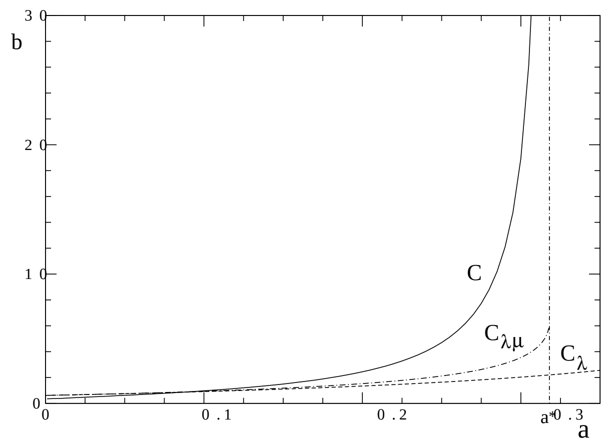

FIG. 4.1. *Curve of the fold limit cycle bifurcation* $C = \{b_f(a),\ 0 \le a < a^*\}$ *so that there are two periodic solutions if and only if* $b > b_f(a)$ *with* $a < a^*$. *Parameters are* $\lambda = 1$, $\mu = 1$. *Curves* C_λ *and* $C_{\lambda\mu}$ *correspond to the two approximations given by* (6.7).

as the expansion of x (it is easy to show that the zero order term is zero). The leading order expansion of $F(x, y)$ is

$$F(x, y) = a(\cos x_1 - 1)r + O(1),$$

which gives us the approximation

$$T = \frac{4\pi}{r} + O\left(\frac{1}{r^2}\right).$$

One notes the similarity with the expression (4.10) obtained above. In the same way, if we write

$$y = \frac{y_1}{r} + O\left(\frac{1}{r^2}\right),$$

the determination of y_1 is carried out by canceling the higher order term of the expansion of $F(x, y)$. We find

$$2\pi \sin(y_1) - \frac{a}{2}x_2^2 - a\pi^2 = 0, \tag{4.12}$$

and it should be noted that the expansion of $F(x, y-x)$ leads to the same expression. We show that $x_2 = 0$ (while remarking that x_2 is the first order term in the expansion of T as $b \to 0$ and using the symmetries of the differential equations (1.5)). We therefore find a solution of (4.12) if and only if

$$a \le \frac{1}{\pi}, \tag{4.13}$$

which enables us to obtain the value $a^* = \frac{1}{\pi}$ (Figure 4.1). When $a < a^*$ and r is large enough, we found two values of τ corresponding to exactly two limit cycles. We remark that, asymptotically, both these cycles have the same period.

We can now predict the behavior of system (1.5) for any (λ, μ). The change of variables

$$(\tilde{b}, \tilde{a}, \tilde{t}, \tilde{v}, \tilde{w}) = \left(\frac{b}{\lambda^2}, \frac{\lambda}{\mu} a, \lambda t, \frac{\lambda}{\mu} v, \frac{w}{\mu} \right)$$

enables us to find the case previously studied. Condition (4.13) is then written as

$$a \leq \frac{\mu}{\lambda \pi}, \tag{4.14}$$

and the bifurcation curve obtained from b_f is given by

$$b = \lambda^2 b_f \left(\frac{\lambda}{\mu} a \right). \tag{4.15}$$

Thus, for sufficiently small λ, the system has always a limit cycle. We will discuss this point in more detail in section 6. For μ large enough, the condition for existence of a periodic solution is written as $b \geq \frac{\lambda^2}{4}$.

5. Geometrical study. We are going to specify the dynamical behavior of the system in the phase plane. We also will make use of a geometrical analysis to characterize solutions of the transcendental equations system (4.7) in the sense that the search for periodic solutions should be carried out among the intersection points of the two curves $C_1 = \{(x, y), F(x, y) = 0\}$ and $C_2 = \{(x, y), F(x, y - x) = 0\}$ in the space region $0 < y < x$. Three configurations can then be distinguished.

5.1. No periodic solution. The simplest situation is obtained when the two curves C_1 and C_2 do not present any intersections. In this case, the origin is globally attractive. We already have illustrated such a configuration in Figure 3.2 and have shown that this case still appears when $\lambda^2 > 4b$ (in this case, C_1 and C_2 are defined using F given by (4.9)).

5.2. Pair of admissible solutions. We have seen that when $r \to +\infty$, it is possible to find exactly two pairs of solutions for system (4.7). When these solutions lead to an expression for the limit cycle, given by (4.4), satisfying the hypotheses (4.2), they correspond to solutions that are admissible. From numerical simulations, it appears that this situation occurs when C_1 and C_2 are two closed convex curves (in the region of the plane where $0 < y < x$). In this case, there are exactly two intersection points, which correspond to the two limit cycles (stable and unstable).

This configuration can also be found using perturbation methods. In particular, we will show a mechanism for the birth of these two limit cycles in the phase plane. We consider the following Hamiltonian system:

$$\begin{aligned} \frac{du}{dt} &= -w, \\ \frac{dw}{dt} &= bv \end{aligned} \tag{5.1}$$

for which the Hamiltonian function, written H, is given by

$$H(v, w) = v^2 + \frac{1}{b} w^2. \tag{5.2}$$

System (1.5) can be written as a perturbation of the Hamiltonian system (5.1)

$$\begin{aligned}\frac{dv}{dt} &= -w - \lambda g(v),\\ \frac{dw}{dt} &= bv,\end{aligned} \tag{5.3}$$

where $\lambda \ll 1$ and $g(v) = v - h(v-a)$. We take $\mu = \lambda$, but we have seen that the study can easily be extended to any (λ, μ) of the same order. It is easy to see that for system (5.3) the origin becomes a focus. In order to have a closer look at what becomes of the periodic trajectory of the center, we are going to use the Melnikov method [11]. This method, which arises from the averaging method, enables us to determine the periodic trajectories that are transformed into limit cycles and thus obtain an approximation of these cycles. The Melnikov function, associated with the level curve $H(v,w) = v^2 + \frac{1}{b}w^2 = l^2$, is

$$M(l) = \int_0^{2\pi} dt\ vg(v)|_{v=l\cos t}.$$

We obtain

$$M(l) = \pi l^2 - 2h(l-a)\sqrt{l^2 - a^2}.$$

Level curves of the unperturbed Hamiltonian system which transform into limit cycles are obtained as solutions of $M(l) = 0$. When $l < a$, the only solution is $l = 0$ and we find that the trajectories tend towards the origin. When $l > a$, $M(l) = 0$ is written as

$$l^4 - \frac{4}{\pi^2}l^2 + \frac{4}{\pi^2}a^2 = 0.$$

There are solutions if and only if $a \le \frac{1}{\pi}$. We then have the following result:

– If $a = \frac{1}{\pi}$, there is a single limit cycle which corresponds to the level curve defined by

$$H(v,w) = \frac{2}{\pi^2}. \tag{5.4}$$

– If $a < \frac{1}{\pi}$, there are two limit cycles which correspond to the level curves defined by

$$H(v,w) = \frac{2}{\pi^2}\left(1 \pm \sqrt{1 - \pi^2 a^2}\right). \tag{5.5}$$

These results can be added to by using system (4.7). When $\mu = \lambda \ll 1$, the second order asymptotic expansion of F gives

$$\begin{aligned}F(x,y) = {} & 2a\sqrt{b}(\cos(2\sqrt{b}x) - 1)\\ & + \Big(- a\sqrt{b}(x^2+y^2) + \frac{a}{4\sqrt{b}}(1 - \cos(2\sqrt{b}x))\\ & + \left(\frac{ax}{2} - y\right)\sin(2\sqrt{b}x) + x\sin(2\sqrt{b}y) + a\sqrt{b}y^2\cos(2\sqrt{b}y)\Big)\lambda^2 + O(\lambda^3).\end{aligned}$$

Canceling the zero order term gives

$$T = \frac{2\pi}{\sqrt{b}} + O(\lambda).$$

To find a zero order approximation of τ, written τ_0, it is necessary to use the second order expansion of F. In this case, the first order term of the expansion of T, written as T_1, is involved and the cancellation of the second order term of F (or of $F(x, y-x)$) is written as

$$-ab^2T_1^2 - a\pi^2 + \pi\sin(2\sqrt{b}\tau_0) = 0.$$

Note that the system obtained by the transformation $\lambda \to -\lambda$ and $t \to -t$ has a phase portrait that is obtained from the original system taking the symmetric with respect to the line $w = 0$. We then have $T(-\lambda) = T(\lambda)$, $\tau(-\lambda) = \tau(\lambda)$ and the expansion of T and τ have the form

$$T = T_0 + T_2\lambda^2 + T_4\lambda^4 + \cdots,$$
$$\tau = \tau_0 + \tau_2\lambda^2 + \tau_4\lambda^4 + \cdots.$$

In particular, we have $T_1 = 0$ and, when $a \leq \frac{1}{\pi}$, we find two possible values for the first term of the expansion of τ corresponding to the two limit cycles obtained above:

$${}^1\tau_0 = \frac{1}{2\sqrt{b}}\arcsin(a\pi),$$
$${}^2\tau_0 = \frac{1}{2\sqrt{b}}(\pi - \arcsin(a\pi)).$$

It is possible to obtain more refined approximations by continuing the series expansion of T and τ using (4.7). The approximation of limit cycles is then given from (4.4), (4.5). It is interesting to note the similarity of the expressions obtained here and those obtained for large r. This result is not surprising given the change of variables (1.7). In Figure 5.1 and Figure 5.2, we show a typical configuration under study. The two limit cycles that have just been characterized correspond to solutions in the classical sense in that they satisfy hypothesis (4.2) and can be obtained by our Fourier analysis.

5.3. Only one admissible solution. From numerical simulations, we observe configurations where there is only one admissible solution for system (4.7). This situation does not only appear when there is a single intersection between C_1 and C_2 since, as we have already mentioned as $a \to 0$, there can be several intersections so that only one of which is suitable. Moreover, it is possible to find exactly two intersections between C_1 and C_2 only one of which is suitable. This situation is illustrated in Figure 5.3. We have therefore detected a single limit cycle that appears to be the stable one. Naturally, the unstable cycle still exists and here we talk about a *generalized solution*, insofar as we cannot define it in the classical sense. From numerical simulations, we observe that the appearance of this *generalized solution* corresponds to a bifurcation of curves C_1 or C_2 in that at least one of these two curves no longer corresponds to a single closed curve (see Figure 5.3).

In the phase plane, the study of the vector field enables us to specify the unstable cycle, called a separatrix because it is the boundary between two domains of attraction. We write the coordinate points $(a, -\lambda a)$, (a, y_B) as A and B, respectively,

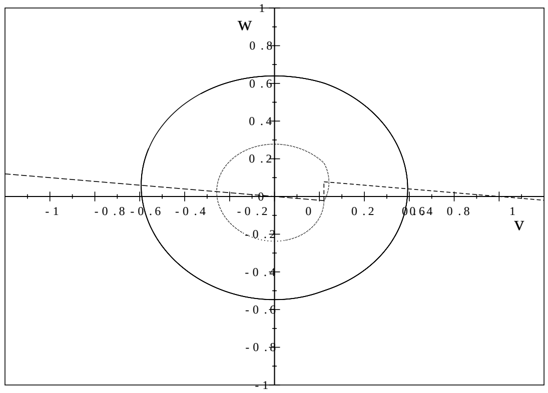

FIG. 5.1. *Unstable (dotted line) and stable (full line) limit cycles of system* (1.5). *The v-nullcline is represented. The parameters are* $\lambda = \mu = 0.1$, $a = 0.22$, $b = 1$.

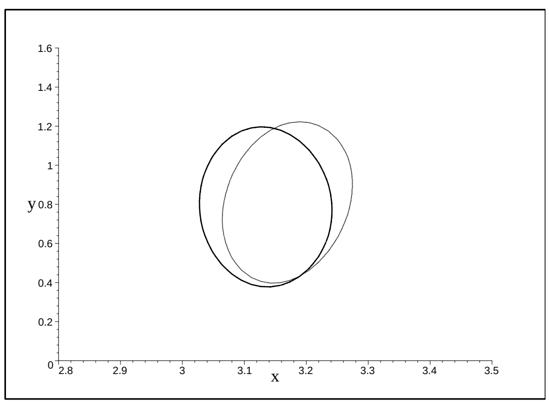

FIG. 5.2. *Curves* C_1 *(thick line) and* C_2 *(thin line). The two intersections correspond to the two limit cycles given in Figure* 5.1 *(the parameters are given in Figure* 5.1*).*

where $y_B \in \mathcal{I}$ and $\mathcal{I}$ designates the interval $[-\lambda a, -\lambda a + \mu]$. Let $\mathcal{P}$ be the parameterized curve obtained when considering the solution of system (1.5) starting from A by reversing the time. The equation for this curve is given by

$$x(t) = a(\cos rt - \tfrac{\lambda}{r}\sin rt)e^{\lambda t},$$
$$y(t) = -a\left(\lambda\cos rt + \tfrac{2}{r}(b - \tfrac{\lambda^2}{2})\sin rt\right)e^{\lambda t}$$

as long as $x(t) < a$. Let t^* be the smallest real so that $t^* > 0$ and $x(t^*) = a$. If $y(t^*) \leq -\lambda a + \mu$, then we take $y_B = y(t^*)$ and the curve $\Gamma = [A, B] \cup \mathcal{P}$ is the boundary being looked for. This situation is displayed in Figure 5.4. If we now have $y(t^*) > -\lambda a + \mu$, we again consider the solution of system (1.5) by reversing the time but with $(x(t^*), y(t^*))$ as the initial condition. This solution crosses the segment $\mathcal{I}$ at the point B that is looked for. If this solution does not present an intersection with $\mathcal{I}$, we are in the presence of an unstable cycle that can be defined in a classical sense given by the resolution of (4.7). Nevertheless, we have not succeeded in establishing

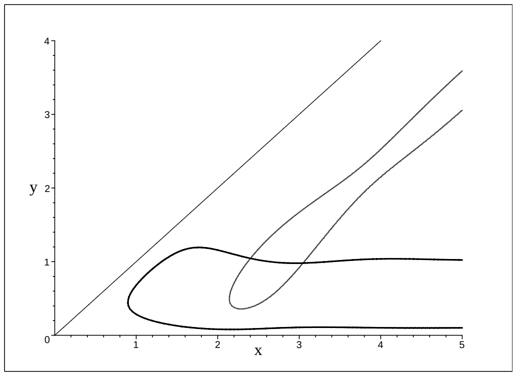

FIG. 5.3. *Curves C_1 (thick line) and C_2 (thin line). The line $y = x$ is represented. The parameters are those of Figure* 5.4.

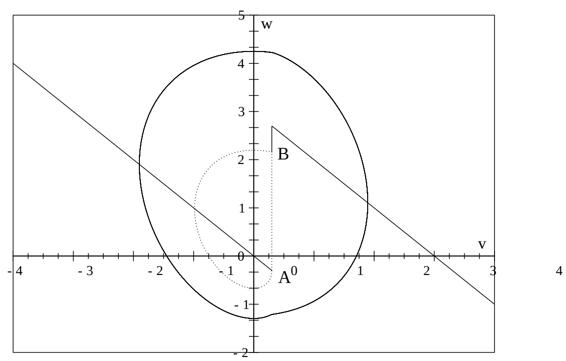

FIG. 5.4. *Stable limit cycle (full line) and unstable limit cycle (dotted line) marking the boundary with the domain of attraction of* $(0, 0)$. *The parameters are $\lambda = 1$, $a = 0.3$, $b = 2$, and $\mu = 3$.*

precise links between the existence of the point B and the solutions of (4.7).

Another approach is to consider a family of *near* systems, the solutions of those tending towards those of (1.5). From this technique arises the mathematical difficulty of the notion of limit being considered. However, let us define the system

$$\begin{aligned} \frac{dv}{dt} &= p_\delta(v) - w, \\ \frac{dw}{dt} &= bv, \end{aligned} \tag{5.6}$$

where $p_\delta(x) = -\lambda x + \mu h_\delta(x - a)$ and h_δ is the continuous function defined by

$$h_\delta(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \frac{x}{\delta} & \text{if } 0 < x < \delta, \\ 1 & \text{if } x \geq \delta. \end{cases}$$

Numerically speaking, for small values of δ, the orbits of (5.6) are a good approximation of those of system (1.5). This result requires careful study, which we have not undertaken here. The convenience of (5.6) is that they allow the application of classical theorems of existence as well as the usual numerical integration methods like the

Runge–Kutta method. In addition, it seems possible to extend the results obtained for the discontinuous system to these continuous piecewise linear systems as $\delta \to 0$.

6. Large relaxation time. In this section, we study the case of a small λ which corresponds to a system with a large time constant. When $\lambda \ll 1$, the asymptotic expansion of (4.8) is written as

$$F(x,y) = -2a\sqrt{b}(1-\cos(2\sqrt{b}x)) + \lambda\mu(x\sin(2\sqrt{b}y) - y\sin(2\sqrt{b}x)) + O(\lambda^2). \tag{6.1}$$

Therefore we have

$$T = \frac{2\pi}{\sqrt{b}} + O(\lambda),$$
$$\tau = \frac{\pi}{2\sqrt{b}} + O(\lambda).$$

The existence of a periodic solution for small λ has already been noted in section 4. We obtain a single solution for τ which is related to the big cycle. The small cycle cannot be captured by this limiting situation. Using the third order expansion of $F(x,y)$ and $F(x,y-x)$, we find that

$$\begin{aligned} T &= \frac{2\pi}{\sqrt{b}} + \frac{\pi}{4b\sqrt{b}}\lambda^2 + O(\lambda^3), \\ \tau &= \frac{\pi}{2\sqrt{b}} - \frac{a\pi}{2\mu\sqrt{b}}\lambda + \frac{\pi}{16b\sqrt{b}}\lambda^2 + O(\lambda^3). \end{aligned} \tag{6.2}$$

Using the first order expansion of T, we calculate

$$f(t) = -\frac{2\mu}{\lambda\pi}\sin\sqrt{b}t + O(1).$$

Calculation of the approximation of v, using (4.5), (4.4), (6.2), gives

$$v(t) = \frac{2\mu}{\lambda\pi}\cos\sqrt{b}t + O(1). \tag{6.3}$$

The approximation that is obtained coincides with the term carrying the fundamental frequency in the Fourier series of v. Using $w_0 = \frac{\mu}{2}$, the limit cycle approximation is given by

$$v^2 + \frac{1}{b}\left(w - \frac{\mu}{2}\right)^2 = \frac{4\mu^2}{\lambda^2\pi^2} + O(1). \tag{6.4}$$

Numerically speaking, this approximation appears to be a good one, even for large values of λ. It is possible to refine the approximation obtained by using higher order terms in the expansion (6.2). We then find

$$v(t) = \frac{2\mu}{\lambda\pi}\cos\sqrt{b}t + \frac{\mu}{\pi}\left(\frac{\pi}{\sqrt{b}} - t\right)\cos\sqrt{b}t + O(\lambda). \tag{6.5}$$

Remark 1. The terms in the expansion of v have zero mean value.

Remark 2. Approximation (6.5) must be considered for $t \in [0,T]$. This raises the problem of matching at T, a problem that we will not discuss here since we will use

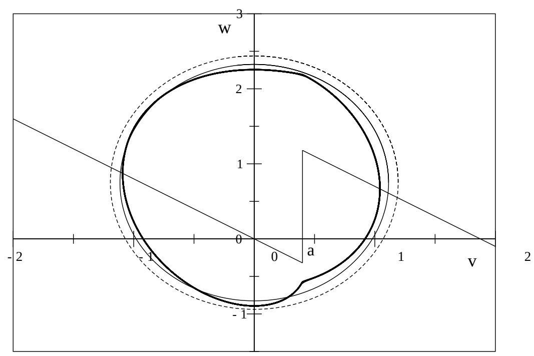

FIG. 6.1. *Stable limit cycle of system* (1.5) *(thick line) and its approximations given by* (6.6) *(thin line) and* (6.4) *(dotted line). The parameters are* $\lambda = 0.8$*,* $\mu = 1.5$*,* $a = 0.4$*, and* $b = 2$*.*

approximation (6.3). From a numerical point of view, this approximation appears to be better for a wide range of values of λ. This is due to the appearance of secular terms in the asymptotic expansion (6.5).

Remark 3. If the expansion of T is continued, there is no term of third order, which leads us to believe that T presents an even power series expansion.

It is interesting to compare the approximation that has just been calculated with the one previously obtained (5.5). The value found for the largest cycle, in the case of small (λ, μ), gives

$$v^2 + \frac{1}{b}\left(w - \frac{\mu}{2}\right)^2 = \frac{2\mu^2}{\lambda^2\pi^2}\left(1 + \sqrt{1 - \frac{\lambda^2\pi^2 a^2}{\mu^2}}\right), \tag{6.6}$$

which, for small values of a, corresponds to the approximation (6.4). Numerically speaking, this approximation is very precise, as shown in Figure 6.1.

Using approximations (6.4) and (6.6), we can formulate an approximate necessary condition for the existence of a periodic solution since the expression of the vector field requires that the interior of the limit cycle contains the point $(a, -\lambda a)$, which yields to

$$b > \frac{(\lambda a + \frac{\mu}{2})^2}{d^2 - a^2} \tag{6.7}$$

with $d \in \{d_\lambda, d_{\lambda\mu}\}$, where d_λ^2 and $d_{\lambda\mu}^2$ are the values of the right-hand term of equations (6.4) and (6.6), respectively. Approximation (6.4) imposes the condition $a < \frac{2\mu}{\lambda\pi}$, which is a requirement greater than that given by (6.6). Even far from its validity domain, approximation (6.7) remains useful. When $\lambda = 1$ and $\mu = 1$, Figure 4.1 shows the approximation (6.7) obtained from the study for small λ (curve C_λ) and for small (λ, μ) (curve $C_{\lambda\mu}$). For small values of a, the requirement appears to be a little too strong, in that it imposes $b > \frac{\pi^2}{16}$ when $b > \frac{1}{4}$ would do.

Let us precisely give the bifurcation giving rise to the stable limit cycle for small λ. In this case, the system under study may be considered as a perturbation of

$$\frac{dv}{dt} = \mu h(v - a) - w,$$

(6.8)
$$\frac{dw}{dt} = bv.$$

System (6.8) was previously considered as a perturbation of the Hamiltonian system obtained for $\mu = 0$. However, in this analysis, μ is not considered as a small parameter. The harmonic balance method leads to the following two cases:

- τ does not exist and we find a family of periodic solutions defined by

  (6.9) $$H(v, w) = c^2, \quad \text{where} \quad c < a,$$

  where H is given by (5.2).
- If we assume that τ exists, we find that the Fourier series expansion of v is divergent and therefore there is no periodic solution such as $v > a$.

For an initial condition outside the ellipse obtained with $c = a$ in (6.9), a solution of (6.8) tends towards infinity since the orbits of system (6.8) are given by

$$v^2 + \frac{1}{b}w^2 = \text{const} \quad \text{for} \quad v < a,$$
$$v^2 + \frac{1}{b}(w - \mu)^2 = \text{const} \quad \text{for} \quad v > a,$$

and, if we consider the sequence $(w_n)_{n\in\mathbb{N}}$ associated with the Poincaré section defined by $v = a$, we have

$$w_n = w_{n-1} + 2\mu.$$

Thus, the orbits spiral around the origin and move away from it. The addition of the perturbation $-\lambda v$ leads to (i) the destruction of the family of periodic solutions so that $v < a$ (the origin becomes a stable focus) and (ii) the appearance of a limit cycle towards which the orbits converge while spiraling. We have seen that the birth of the limit cycle takes place at ∞ since the diameter of the ellipse can be made arbitrarily large. We are going to specify this result in bifurcation terms.

We write (r, θ) for the polar coordinates of (v, w) and, because we are interested in the system at ∞, we introduce the variable $u = \frac{1}{r}$. Given a change of variables, we can consider the case $b = 1$. Writing (1.5) using the new variables gives

$$\frac{du}{dt} = \lambda u \cos^2\theta - u^2\mu\cos\theta h\left(\frac{\cos\theta}{u} - a\right),$$
$$\frac{d\theta}{dt} = 1 + \lambda\sin\theta\cos\theta - u\mu\sin\theta h\left(\frac{\cos\theta}{u} - a\right).$$

We are interested in the behavior of the system for $\lambda \ll 1$ and u close to 0. In this case, θ is a fast variable, the dynamic of which can be approximated by $\theta' = 1$. The averaging theorem [11] enables us to consider the approximation given by the averaged system

$$\frac{du}{dt} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \quad \lambda u\cos^2\theta - u^2\mu\cos\theta h(\cos\theta),$$

where we have used the approximation $h(\frac{\cos\theta}{u} - a) \sim h(\cos\theta)$ for small $u > 0$.

We find

$$\frac{du}{dt} = \left(\frac{\lambda}{2} - \frac{\mu}{\pi}u\right)u,$$

which shows the appearance of a stable limit cycle. The radius of this cycle is given by $u = \frac{\lambda\pi}{2\mu}$ and is in agreement with the approximation (6.4). This is a supercritical Andronov–Hopf bifurcation which appears at ∞. As far as we know, such a bifurcation was mentioned for the first time in [37].

7. Excitability and oscillations. We may interpret the appearance of oscillations as the limit behavior of a *spike solution* when the number of action potentials becomes large. We are going to give mathematical content to this statement by showing that the periodic solution, written as $v_\gamma(t)$, can be obtained as the limit of the *spike solution*, written as $v_n(t)$, when the number of spikes n tends towards $+\infty$. Most often, the birth of oscillations is shown in terms of bifurcations using equations based on system parameters. Here, the characterization is directly obtained from the system solutions.

We consider (3.7), omitting the transient regime containing v_0, because we are interested in the asymptotic state. Using a time shift, we consider the symmetrical sum obtained from (3.7):

$$v_n(t) = \sum_{k=-n}^{n} \phi(t - t_{2k}) - \phi(t - t_{2k+1}),$$

where

$$\phi(t) = h(t)\varphi(t)$$

and φ is given by (3.8). If we assume that the spikes are produced at periodic time intervals, there exist T and τ so that $t_{2k} = kT - \tau$ and $t_{2k+1} = kT + \tau$. The existence of the pair (T, τ) is studied in section 4. We should also note that the assumption just made is linked to v_0 insofar as not all orbits converge towards a periodic solution.

We have $\phi \in L^1(\mathbb{R})$, and the Poisson formula, in the space of tempered distributions $\mathcal{S}'$, gives us

$$\lim_{n\to+\infty} v_n(t) = \frac{1}{T} \sum_{-\infty}^{+\infty} \widehat{\phi}\left(\frac{k}{T}\right) 2\ i\ \sin\left(2\pi\tau\frac{k}{T}\right) e^{2i\pi k\frac{t}{T}}.$$

As ϕ', the distributional derivative of ϕ, is in $L^1(\mathbb{R})$, equality occurs for every t, and we have the uniform convergence of the series. We calculate that

$$\widehat{\phi}(w) = \frac{2\mu}{2b - 8\pi^2 w^2 + 4i\pi\lambda w},$$

giving

$$\lim_{n\to+\infty} v_n(t) = v_\gamma(t),$$

where $v_\gamma(t)$ is the periodic solution given by (4.3), which establishes the stated result.

8. Discussion. Estimation of the maximal number and relative positions of limit cycles of a two-dimensional autonomous system is an open problem corresponding to the second part of the sixteenth Hilbert problem. Given the difficulty of the general problem, mathematicians have become interested in a particular system class, the Liénard system:

$$\text{(8.1)} \qquad \begin{aligned} \frac{dv}{dt} &= p(v) - w, \\ \frac{dw}{dt} &= v. \end{aligned}$$

Most results concern the case in which p is a polynomial function. Even in this case, there are no general theoretical results and most approaches are local ones insofar as they determine only the number of limit cycles for certain parameter values. Limit cycles are obtained using perturbation methods via a Hopf bifurcation or a global bifurcation (see [28] and the references therein). Some global approaches make it possible to link the number of limit cycles to the roots of a polynomial [10], [25], but the results remain to be demonstrated.

We have studied the Liénard system where p is a piecewise linear function (linear on $]-\infty, a[$ and on $]a, +\infty[$) allowing a finite jump discontinuity at $a > 0$. We have shown that the limit cycles are characterized by the roots of a system of two transcendental equations. These roots correspond to the period of the oscillations and to an additional parameter. We have obtained an explicit expression of the limit cycles as a function of these two roots. Our results are in agreement with the local methods in that (i) the fold limit cycle bifurcation can be obtained as a perturbation of a center and (ii) the large size limit cycle can be obtained as a Hopf bifurcation at ∞. We might also consider the limit cycle obtained as $a \to 0$ as a kind of *degenerated* Hopf bifurcation. We have shown the existence of at least two limit cycles, and arguments similar to those used in [27] should enable us to demonstrate that at most two limit cycles exist. When p is a polynomial function, such a result can be obtained only for a polynomial of degree at least five [36]. It has already been observed that discontinuous dynamical systems have a richer dynamic than regular dynamical systems [9]. The obtained results, and numerical simulations that we have carried out, lead us to formulate two conjectures concerning the number of limit cycles of a piecewise linear Liénard system.

Conjecture 1. The Liénard system (8.1), with p piecewise linear on $n+1$ intervals and having n finite jump discontinuity, has up to $2n$ limit cycles.

Conjecture 2. The Liénard system (8.1), with p continuous and piecewise linear on $n+1$ intervals, has up to n limit cycles.

Conjecture 2 generalizes the result obtained in [26], [27] in which the authors proposed a continuous, and piecewise linear on $2n+1$ intervals, function p so that Liénard system (8.1) has exactly n limit cycles. The parity and periodicity of p appear to be the two properties that limit the number of limit cycles.

Beyond mathematical interest of the system under study, it is of great importance in mathematical biology where excitable systems are widely used [31], [17]. Our system is a piecewise linear version of the FitzHugh–Nagumo equations with a simplified version of the recovery process which provides an understanding of the behavior in a transparent way. First of all, we have distinguished between two dynamics according to the value of $\lambda^2 - 4b$. When $\lambda^2 - 4b \geq 0$, the system is termed *leaky integrator* and only a single spike can be emitted in response to an excitation given by the input $I = I_0\delta(t - t_0)$. When $\lambda^2 - 4b < 0$, the system is referred as being *resonator*. In this case, the response is obtained as the superposition of

$$v(t) = e^{-\frac{t}{\eta}} \sin \Omega t,$$

where $\eta = \frac{1}{\lambda}$ and $\Omega = \sqrt{b - \frac{\lambda^2}{4}}$ denote, respectively, the time constant and the natural frequency of the system. When this response is a finite sum, we obtain what we call a *spike solution*. In the case of infinite sum, we obtain a periodic solution for which an analytical expression is given by

$$v = \frac{1}{2}(S_{-\tau}f - S_{\tau}f),$$

where S is the shift operator $S_y g(x) = g(x+y)$ and f is a function that depends on the period T. In the general case, it is not possible to have an explicit expression for T and τ. However, we have obtained a set of approximate solutions which shows that the period is well approximated using

$$T = \frac{4\pi}{\sqrt{4b - \lambda^2}}.$$

We have detected two possible mechanisms for the appearance of oscillations: a fold limit cycle bifurcation and a Hopf bifurcation at infinity.

A significant biological interest is the extension of our analysis to the complete system where the recovery process is given by

$$\frac{dw}{dt} = b(v - \gamma w).$$

In this case, a change of variables allows us to rewrite the FitzHugh–Nagumo system as the generalized Liénard equation

$$\begin{aligned} \frac{dv}{dt} &= F(v) - w, \\ \frac{dw}{dt} &= G(v). \end{aligned} \tag{8.2}$$

When p is the polynomial function (1.2), the two functions F and G are third degree polynomial functions and, in contrast to the case $\gamma = 0$, three limit cycles can be obtained. We plan to explore the piecewise linear case for which an analytical study is possible but yields much more complicated expressions than those obtained in this paper. Results on such an extension will be reported elsewhere.

There remains much work to be done on our system. The simplicity of the model allows us to hope for analytical results for bursting [35]. The coexistence of a limit cycle and a stable fixed point favors the existence of such a phenomenon when an additional slow variable is added to the system. Another aspect is the study of coupled equations. In particular, we hope for promising results concerning the dynamics of coupled oscillators using the approximations obtained for the periodic solution. As a first step, we plan to explore the forced system in the context of forced piecewise linear systems [4], [21].

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