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CONCURRENT MULTIPLE IMPACTS IN RIGID BODIES: FORMULATION AND SIMULATION.

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Abstract
The analysis of collisions in multi-body systems has been a topic of continuous research in recent years. The purpose of this paper is to exhibit a multiple impact law for rigid body dynamical system built on physically motivated parameters. This law must meet the properties of closing the non-smooth dynamical equations and of corroborating experiments. Studying the most basic multi-body impact problem (the chain of ball), we highlight the properties of our *impulse ratio* based law. These properties are : existence and uniqueness of the post-impact velocities, compliance with the unilateral constraints, energy balance and the numerical tractability. We then attempt to extend our formulation to the general case of Lagrangian systems by studying a bi-dimensional chain. This study paves the way towards more general results on the quantitative and qualitative properties of a multiple impact law based on a impulse ratio.

Key words
Non-smooth dynamics, multiple impacts, rigid bodies, Newton's cradle, unilateral constraints.

1 Introduction

Position of the problem. Concurrent multiple impacts in rigid bodies can be defined as the occurrence of several impacts at different points of the system at the same time. Such impacts in a mechanism composed of rigid bodies is a common feature of many practical machines and multi-body systems, such as the mechanics of circuits breakers, granular materials, robot systems, *etc*. In order to correctly simulate and design these systems, impacts must be modeled correctly. Due to the complexity of the task, we will focus in this paper on simple examples where configuration is straightforward : the chain of balls or Newton's cradle are academic example where concurrent multiple impacts occur.

In a classical single impact case, the definition of an impact law allows one to compute the post impact velocity in a unique way with some mathematical cares on data (Glocker, 2001). Most of the classical formulations fall short of modeling multiple impacts. A reliable multiple impact law must feature the following properties:

1. Uniqueness and existence : given the data of the problem (pre-impact velocities, present configuration, systems parameters), the impact law has to provide us with a unique solution.
2. This solution has to satisfy the equations of motion, the unilateral constraints on velocities and impulses, and finally the energy balance.
3. In terms of experimental observations, this solution has to corroborate them and also be constituted of measurable and physically justified set of parameters.
4. Last but not least, the law must be numerically tractable. A formulation in terms of a non-smooth minimization problem or a complementarity problem can help in achieving this point.

Background. A large number of articles are devoted to the study of particular type of systems where some multiple impacts occur, particularly in the community of physicists, see for example (Herrmann and Seitz, 1982). However a correct modeling of such phenomena still lacks. An attempt has been made by Han and Gilmore(Han and Gilmore, 1993), to propose a sequential approach of this problem. This method is an analytical computer-oriented method that results in either several sets of feasible post-impact velocities or physically meaningless post-impact dynamics. In fact the authors assume the possibility of a certain chronology for the possible impacts occurring in the system, hence sequential impacts. The main drawback is the lack of mathematical results on the uniqueness and existence of solutions. As we will see in the 3-ball chain example, it's a priori impossible to decide which one is the right solution in the set of feasible solutions. Motivated by an experimental work on Newton's cradle, Ceanga and Hurmuzlu (Ceanga and Hurmuzlu, 2001) postulate the existence of an *Impulse Correlation Ratio* (ICR) for a triplet of balls. They succeeded in computing the coefficient of restitution and the impulse correlation ratio that produced the best fit to the experimentally acquired post impact velocity. In this work, the same remark as above may be made. The lack of mathematical results and also of a clear algorithm to find the solution are the major drawbacks.

On a more theoretical point of view, the works of Frémond (Frémond, 1995) and Glocker *et al.*(Glocker, 2001; Aeberhard and Glocker, 2005) lead to two rigorous frameworks for formulating a general impact theory, imposing several conditions to obtain an impact law consistent with the fundamental principles of Thermodynamics and Mechanics. It is clear that the new developments of impact laws, even for multiple impacts must lie into these contexts. The only drawback is that a rigorous framework does not give the precise formulation of a law but just a canvas. Particularly, a precise physical definition of the parameters somewhat lacks.

Finally, the work presented in this article is the continuation of the work of Acary and Brogliato (Acary and Brogliato, 2003*a*; Acary and Brogliato, 2003*b*) where the interest of an impulse ratio between each couple of constraints has been highlighted on examples of chains of balls. On a mono-dimensional example, this set of parameters together with a global dissipation coefficient lead to a right parameterization of the post impact velocities. Furthermore, in the case of balls, where Hertz contact is suitable for the modeling of the compliant and viscous behavior at contact, this set of impulse parameters are easily evaluated and their physical meanings are quite clear. It's noteworthy that this impulse ratio is completely different from the ICR introduced by Ceanga and Hurmuzlu.

An outline of the article. The aim of this work is to exhibit a multiple impact law that meets the four preceding conditions. Drawing our inspiration from the aforementioned work, we use an impulse ratio based law and attempt to extend it to more general configuration for rigid bodies in the case of Lagrangian systems.

This paper is organized as follows: in Section 2, we state briefly the basic equations for a Lagrangian systems with impact. Particularly, we review some simple and multiple impact laws. In the Section 3, we recall some results on the application of the impulse ratio on a monodimensional chain of balls. In the section 4, in order to extend to a bidimensional system, we study the multiple impact of balls in a plane. For 3 balls, we highlight the role of the kinetic angle between two constraints and the angle of incidence of the first ball. Some more general interpretation of the role a such angles are given in the Section 5.

2 Lagrangian systems with unilateral constraints and impacts

In this section, we briefly present standard ingredients of the finite-dimensional dynamics of rigid bodies with unilateral constraints and possibly impacts. For more details, we refer to (Glocker, 2001; Glocker, 2004) and (Brogliato, 1999).

2.1 Dynamics and unilateral constraints

We take advantage of the Lagrangian formulation of the dynamics, with the generalized coordinates denoted by $q(t) : [0,T] \mapsto \mathbb{R}^n$, $M(q)$ the mass matrix and $F(q,\dot{q},t)$ is the set of forces acting upon the system.

Let us consider a system with ν unilateral constraints $h_\alpha(q)$ defined by the following set of ν inequalities :

$$h_\alpha(q,t) \geq 0 \qquad \alpha = 1\ldots\nu, \tag{1}$$

The unilateral constraints determine the set of admissible configuration $\mathcal{C}$ as :

$$\mathcal{C}(t) = \{q \in \mathcal{M}(t) | h_\alpha(q,t) \geq 0\} \tag{2}$$

where $\mathcal{M}$ is the configuration manifold. The Lagrangian equations of the motion on the interval with sufficient regularity are :

$$M(q)\ddot{q} + F(q,\dot{q},t) = \sum_{\alpha=1}^{\nu} \nabla h_\alpha(q)\lambda_\alpha, \tag{3}$$

where λ_α is the set of the Lagrange multipliers associated with the constraints $h_\alpha(q,t)$ through a complementarity condition :

$$0 \leq h_\alpha(q,t) \perp \lambda_\alpha \geq 0, \tag{4}$$

This relation prevents from inconsistent results that allows the penetration of the bodies in each other.

If the evolution is no longer smooth , we enter in the field of the Non smooth dynamics where the equations of motion must be written in terms of real measures :

$$M(q)du + F(q,u,t)dt = \sum_{i=1}^{m} \nabla h_\alpha(q)\lambda_\alpha, \tag{5}$$

where du is the differential measure associated with the function of bounded variations $u(t) = \dot{q}^+(t)$, dt the Lebesgue measure and λ_i is hence the force with a positive real measure.

At an instant of impact, t_k, (5) reads as:

$$\begin{cases} M(q_{t_k})\Delta\dot{q} = P \\ v = H\dot{q} \\ P = H^T p \\ H = \nabla h(q(t_k)) \end{cases} \tag{6}$$

where $\Delta\dot{q} = \dot{q}(t_k^+) - \dot{q}(t_k^-)$ is the jump in the generalized velocity due to impact and $P \in \mathbb{R}^n$ may be defined as the generalized impulse forces. The vectors $v \in \mathbb{R}^\nu$ and $p \in \mathbb{R}^\nu$ collect the components of the normal relative velocity v_α and the normal impulse p_α which is only the amplitude of the delta measure in λ_α. This set of Dynamical equations and kinematic relations may be condensed in

$$\Delta v = H\, M(q_{t_k})^{-1}\, H^T p \tag{7}$$

where $\Delta v = v(t_k^+) - v(t_k^-)$ is the jump in the local velocity.

The perfect unilateral constraints require that :

$$\text{if } h_\alpha(q(t_k)) = 0 \text{ then } \begin{cases} p_\alpha \geq 0 \\ v_\alpha(t_k^+) \geq 0 \end{cases} \quad \forall\alpha = 1\ldots\nu \tag{8}$$

2.2 Standard impact laws

Clearly, the set of equations (7) and (8) is not sufficient to define a well posed problem. We need an additional information on the behavior of the system at impact. This information is usually provided by the impact law. We recall in this section the formulation of the most well-known laws.

Newton restitution coefficient Newton postulated a ratio between the relative post and pre-impact velocities as

$$v_\alpha(t^+) = -e v_\alpha(t^-), \forall\alpha = 1\ldots\nu \tag{9}$$

A more general formulation prone to numerical and mathematical analysis has been given by Moreau as

$$0 \leq v(t^+) + e v(t^-) \perp p \geq 0 \tag{10}$$

where the complementarity relation has to be taken component-wise.

Poisson restitution coefficient This dynamical approach consists in separating the impact process into two phases : a compressive phase followed by the energy release phase called the restitution phase. The Poisson's coefficient reads :

$$e_p = \frac{p_{r\alpha}}{p_{c\alpha}} \tag{11}$$

where $p_{c\alpha}$ and $p_{r\alpha}$ are respectively the impulses at the end of the compression and restitution phases at the constraint α.

Glocker and Pfeiffer (Glocker and Pfeiffer, 1995) have proposed a restitution mapping to calculate the post impact velocity in the Lagrangian setting with a complementarity formulation. This formulation allows us to derive mathematical results and efficient numerical algorithm.

Global Energetic Coefficient A natural way to define globally a coefficient of restitution is to define a ratio on the pre and post-impact kinetic energy :

$$\frac{1}{2}\dot{q}(t^+)^T M \dot{q}(t^+) = e\dot{q}(t^-)^T M \dot{q}(t^-) \tag{12}$$

The advantage of this kind of coefficient is that it ensures the energetic balance contrary to both aforementioned cases . The drawback is that we have to deal with a quadratic equation in terms of the generalized velocities.

3 Mono-dimensional chains of balls

3.1 Case study of three balls chain

In this section, we focus our attention on 3-ball chains, which are very interesting examples of systems with multiple impacts. We recall classical results on the application of the standard impact laws and following the work in (Acary and Brogliato, 2003*b*), we show the pertinence of introducing an impulse ratio.

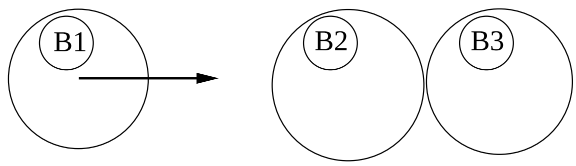

Figure 1. chain of hard balls

3.2 Rigid body models of a 3-ball chain

A dynamical system of three rigid balls of equal mass m, described by their center of mass positions q_1, q_2, q_3 and velocities $\dot{q}_1, \dot{q}_2, \dot{q}_3$ is considered. We assume the normal collision on a straight line without friction. The dynamics at the instant of impact is:

$$\left.\begin{array}{l} m(\dot{q}_1^+ - \dot{q}_1) = -p_1 \\ m(\dot{q}_2^+ - \dot{q}_2) = -p_2 + p_1 \\ m(\dot{q}_3^+ - \dot{q}_3) = p_2 \end{array}\right\} \tag{13}$$

The perfect unilateral constraints require that :

$$\left.\begin{array}{l} \dot{q}_1^+ \leq \dot{q}_2^+ \leq \dot{q}_3^+ \\ p_1 \geq 0, p_2 \geq 0 \end{array}\right\}, \tag{14}$$

where $\dot{q}_i$ and $\dot{q}_i^+$ are respectively the pre-impact and post-impact velocities and p_j the impulses.

Newton restitution coefficient If we choose a particular case where $e_1 = e_2 = \dot{q}_1 = 1$ and $\dot{q}_2 = \dot{q}_3 = 0$, the solution is the following : $[\dot{q}_1^+, \dot{q}_2^+, \dot{q}_3^+] = [-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}]$ and $[p_1, p_2] = [\frac{4}{3}, \frac{2}{3}]$. Practically, this solution means that the ball 1 impacts the two stationary balls and bounces back at the velocity $\frac{1}{3}$ and the balls 2 and 3 stick together to move at the speed $\frac{2}{3}$. This solution is neither quantitatively nor qualitatively acceptable. The next solution to explore is the Poisson's restitution.

Poisson restitution coefficient The compression percussions p_{1c} and p_{2c} are calculated thanks to $\dot{q}_1^+ = \dot{q}_2^+ = \dot{q}_3^+$. Finally we have $p_1 = p_{1c}(e_{p1} + 1)$ and $p_2 = p_{2c}(e_{p2} + 1)$. We deduce from this the same solution yielded by the Newton's approach, i.e. $[\dot{q}_1^+, \dot{q}_2^+, \dot{q}_3^+] = [-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}]$ ....

Sequential impact With the method proposed by Han and Gilmore, and with the assumption of the conservation of Kinetic energy, two post-impact velocities are computed and given by :

$$(\dot{q}_1^+, \dot{q}_2^+, \dot{q}_3^+) = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \tag{15}$$

$$(\dot{q}_1^+, \dot{q}_2^+, \dot{q}_3^+) = (0, 0, 1) \tag{16}$$

The first solution of their method is common with the aforementioned solutions. We know that this solution is not good from a quantitative and qualitative points of view. The latter solution means that body 1 and 2 remain stuck when body 3 moves to the right. This one is also not good from a qualitative point of view. It's easy to observe that in practice the third ball detaches quickly from the second one, the first and second one possess non-zero post-impact velocity and do have a motion after the collision. This slight motion is quite important to understand what happens asymptotically in the cradle (all the balls utimately stick together and move without impacts, hence the name "cradle").

It's noteworthy that these standard laws yield admissible results in the simple impact case. This shows clearly that multiple impacts are not a trivial extension of simple impacts. The restitution coefficients for simple impacts may completely lose their meaning when passing to multiple shocks. In fact the determination of velocities is trivial, when no contact are closed during an excitation. But if one contact is closed, the problem gets uncomfortable, because the contact can react with impulsive forces of a priori unknown magnitude.

Non local Mechanics Within the theoretical framework of M. Frémond, C. Cholet(Cholet, 1998) proposed a multiple impact law based on a pseudo-potential of dissipation. The percussion and particularly the percussion at distance derives from this pseudo-potential with the respect to the relative velocity. The only drawback of the approach is that the solution is formulated in terms of three parameters with no obvious physical meanings.

3.3 Global dissipation coefficient and impulse parameter

We recall in this section the results of (Acary and Brogliato, 2003*b*). We use the impulse ratio denoted by $\alpha = p_2/p_1$ whose physical meaning will be explained in the next section. This relation appears to be crucial to the well posedness of the system . The dissipation is taken into account through the equation of the global energy balance (12):

$$\begin{cases} (\dot{q}_1^+)^2 + (\dot{q}_2^+)^2 + (\dot{q}_3^+)^2 = e(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) \\ \alpha = \dfrac{p_2}{p_1} \end{cases} \tag{17}$$

We have 5 unknowns in terms of 3 equations of impact dynamics and 2 equations of energy balance and the *impulse ratio*. The post-impact velocities and the impulses arising from the system of equations (13) -(17) for a given value e and α are the following :

$$\begin{aligned} p_1 &= \frac{(1+\sqrt{\Delta})m}{2(1-\alpha+\alpha^2)} \\ p_2 &= \frac{(1+\sqrt{\Delta})m\alpha}{2(1-\alpha+\alpha^2)} \\ \dot{q}_1^+ &= -\frac{1+\sqrt{\Delta}}{2(1-\alpha+\alpha^2)} + 1 \\ \dot{q}_2^+ &= -\frac{(1+\sqrt{\Delta})(\alpha-1)}{2(1-\alpha+\alpha^2)} \\ \dot{q}_3^+ &= -\frac{(1+\sqrt{\Delta})\alpha}{2(1-\alpha+\alpha^2)} \end{aligned} \tag{18}$$

with $\Delta = -1 + 2e - 2\alpha e + 2\alpha + 2\alpha^2 e - 2\alpha^2$. The positiveness' of Δ and the unilateral constraints lead to the following inequalities :

$$\begin{aligned} &\tfrac{1}{3} \leq e \leq 1 \\ &\tfrac{1}{2} \leq \alpha \leq \frac{2e - 2 - \sqrt{6e-2}}{e-3} \end{aligned} \tag{19}$$

One can easily check that the solution which is $[\dot{q}_1^+, \dot{q}_2^+, \dot{q}_3^+] = [0, 0, 1]$ is encompassed by setting $\alpha = 1$ and $e = 1$.

To reproduce what happen experimentally, we can always find out a perfect couple of α and e that matches the experimental data, and/or the results of numerical experiment with Hertzian–spring and dashpot (see

§ 3.4). For instance, in the case of perfect elastic impact i.e. $e = 1$ with $\alpha = 0.95$, the result, $[\dot{q}_1^+, \dot{q}_2^+, \dot{q}_3^+] = [-0.05, 0.05, 0.99]$ indicates that the ball 1 and 2 are no longer stationary after the impact. Instead they move slightly while the ball 3 moves with almost the same velocity as the initial velocity of ball 1.

3.4 Chain of n-balls regularized with Hertian Visco-elastic contact model.

In order to illustrate the interest for such kind of impulse ratios, Acary and Brogliato have studied a uni-dimensionnal chain of n balls in mulitple impacts. We have followed the results of (Falcon *et al.*, 1998),(Hertzsch *et al.*, 1995) and (Ramirez *et al.*, 1999) showing experimentally and analytically that the visco-elastic Hertzian model of contact

$$f = K\delta^n + C\delta^{n-1}\dot{\delta} \tag{20}$$

is the right one. The indentation δ is then correlated with the effort at contact f in a smooth way. The dynamics of the chain of balls is then regularized and numerically time-integrated with a standard ODE solver. Thanks to this modeling and to the numerical experiments, we are able to state some general results on the impulse ratios and to evaluate them in a straightforward way by :

$$\alpha_{\gamma\beta} = \frac{\displaystyle\int_0^{t_f} f_\gamma(t)\,dt}{\displaystyle\int_0^{t_f} f_\beta(t)\,dt} \tag{21}$$

where t_f corresponds to the end of the multiple impact and β and γ are indexes of constraints.

The major results are :

1. The post-impact velocity, computed with the multiple impact law defined by *impulse ratio*, augmented by a total energetic law (12), is provided in a unique way and the system becomes mathematically well-posed.
2. If the perfect constraints are regularized by a general viscoelastic contact model corresponding to a linear viscoelastic bulk behavior (Hertzsch *et al.*, 1995; Ramirez *et al.*, 1999) (20) as in (20) then
   (a) The ratio of impulse is finite and the subspace of the state space defined by

   $$E = \{\delta \geq 0, \dot{\delta} \geq 0\} \tag{22}$$

   is globally attractive. Moreover, the amplitude of the force asymptotically tends towards zero and the relative velocity $\dot{\delta}$ towards a finite constant. This last point is very important from a numerical point of view. Extending these results to finite-time convergence is still an issue.

   (b) The impulse ratios are independent of the absolute value of stiffness and masses but only function of the ratio of stiffness and mass. This result allows us to consider the rigid case as limiting case in the same way.
3. If the perfect constraints are regularized by a linear model, i.e.

   $$f = K\delta, \tag{23}$$

   we can prove that the instants of changes in the contact interactions, are an adimensional scale of time, for instance, $T = \omega_i t$ (where ω_i is a modal frequency of the system), and the ratio of impulses, α, do not depend on the absolute values of stiffness k and mass m. Moreover, the impulse ratio α, is completely determined by the natural modes of the regularized dynamical system and the pre-impact velocities.

This conclusion outlines two important consequences:

(1) from a mechanical point of view, the introduction of an impulse ratio enhances the model with some informations about the behavior of dynamical system when it is binded by elastic contact,
(2) from a numerical modeling point of view, the independence to absolute value of k allows one to consider in a consistent manner its applications to very large stiffnesses, which are generally encountered in applications.

Other remarks on the physical interpretation of the impulse ratio may be found in (Acary and Brogliato, 2003*b*; Acary and Brogliato, 2003*a*).

4 The pool: A bimensional example

The aim of this section is to extend the results of mono-dimensional case to a bidimensional case in view of a generalization for every finite dimensional Lagrangian system.

4.1 Two stationary balls struck by one ball

Presentation of the problem Let us consider a normal collision without friction of a ball with two stationnary ones depicted in the Figure 2. The balls are identical. We set for the sake of simplicity the mass $m = 1$ and the radius $a = 1$ and the norm of the pre-impact velocity of the ball 1 is equal to 1. The points $G_1(x_1, y_1), G_2(x_2, y_2)$ and $G_3(x_3, y_3)$ are respectively the centers of mass of the ball $1, 2$ and 3. The generalized Lagrangian coodinates vector is $q = [x_1, y_1, x_2, y_2, x_3, y_3]^T \in \mathbb{R}^6$.

We introduce three unilateral constraints such that :

$$\begin{cases} h_1(q) = (x_2 - x_1)^2 + (y_2 - y_1)^2 - 2 \geq 0 \\ h_2(q) = (x_2 - x_3)^2 + (y_3 - y_2)^2 - 2 \geq 0 \\ h_3(q) = (x_3 - x_1)^2 + (y_3 - y_1)^2 - 2 \geq 0 \end{cases} \tag{24}$$

The local impulse vector is denoted by $p = [p_1, p_2, p_3]$ and the relative velocity by $v = [v_1, v_2, v_3]$.

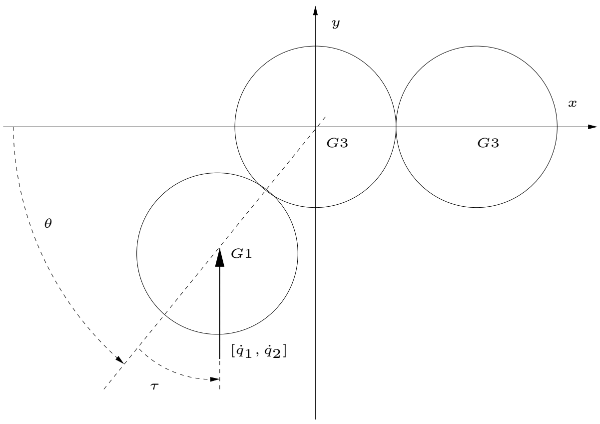

Figure 2. 3-balls in the plane

We use a global energy restitution coefficient (see section 2.2). Applying (12) to our case study gives the following result:

$$\sum_{i=1}^{6} (\dot{q}_i^+)^2 = e((\dot{q}_1^-)^2 + (\dot{q}_2^-)^2) = e \qquad (25)$$

We consider two angles θ and τ to study the post-impact dynamics of the system as depicted in the Figure 2. Due to the symmetry of our problem, we restrict our study to the value of $\theta \in [0, \frac{2\pi}{3}]$ and $\tau \in [0, \pi]$. We are now to search for two or three more equations depending on the value of θ. In fact for $\theta \in [0, \frac{2\pi}{3}[$, we have only two closed contacts so there is no impulse between ball 1 and 3. Therefore the system (7)–(25) have 5 unknowns, hence one additional relations is needed. When $\theta = \frac{2\pi}{3}$, there are three closed contacts so 2 more equations are needed. It is noteworthy that the number of relations to be added depends on the configuration. One goal of this study is to overcome this difficulty by stating a general framework valid for all configurations.

In addition to (25) we use two impulse ratios defined by :

$$\begin{cases} \alpha_{12} = \dfrac{p_2}{p_1} \\ \\ \alpha_{13} = \dfrac{p_3}{p_1} \end{cases} \qquad (26)$$

For the moment, the impulse ratio $\alpha_{13} = 0$ unless $\theta = \frac{2\pi}{3}$. For given values of e and α_{12} and a particular initial velocities $\dot{q} = [-\cos(\theta+\tau), -\sin(\theta+\tau), 0, 0, 0, 0]$ let us study, according to θ and τ, the uniqueness and the existence of solutions of the system (7)– (8)– (25)–(26) after the impact. The aim of the further study is to comprehend the effect of the angle θ and τ on the effectiveness of the impulse ratio. The impulse ratio α_{13} equals zero for $\theta \neq \frac{2\pi}{3}$.

4.1.1 The influence of θ

Let us assume in this section that we are in the case of a normal collision, i.e., $\tau = 0$. The solution of (7)- (25)- (26) for $\tau = 0$ is :

$$\begin{cases} p_1 = \dfrac{1+\sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2} \\ p_2 = \dfrac{(1+\sqrt{\Delta})\alpha_{12}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2} \\ \dot{x}_1^+ = \cos\theta(1 - \dfrac{1+\sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2}) \\ \dot{y}_1^+ = \sin\theta(1 - \dfrac{1+\sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2}) \\ \dot{x}_2^+ = (\cos\theta - \alpha_{12})(\dfrac{1+\sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2}) \\ \dot{y}_2^+ = \sin\theta(\dfrac{1+\sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2}) \\ \dot{x}_3^+ = \alpha_{12}(\dfrac{1+\sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2}) \\ \dot{y}_3^+ = 0 \\ \text{with} \\ \Delta = 2(e-1)(\alpha_{12}^2 - \alpha_{12}\cos\theta + 1) + 1 \end{cases} \qquad (27)$$

The positiveness of Δ and the unilateral constraints require some conditions on α_{12} and e.

Valid set of values of e and α_{12} Let's remind the unilateral constraints (8) apply to our case study.

$$\begin{cases} \Delta \geq 0 \\ p \geq 0 \\ v_i^+ \geq 0 \qquad \forall i = 1..3 \end{cases} \qquad (28)$$

The positiveness of the relative velocity v_2^+ imply that $\forall \theta < \dfrac{\pi}{2}, \alpha_{12} \in \left[\dfrac{\cos\theta}{2}, \infty\right)$. To study the valid set of the couple(α_{12}, e) that comply with last equation in the system (28), we define the function:

$$\begin{array}{rcl} g : [0,1] & \longmapsto & \mathbb{R} \\ e & \longmapsto & \dfrac{(2+\cos^2\theta)\alpha_{12}^2 - 4\cos(\theta)\alpha_{12} + 2}{\cos^2\theta\alpha_{12}^2 - 4\cos(\theta)\alpha_{12} + 4} \end{array} \qquad (29)$$

such that $v_1^+ = 0, \forall e = g(\alpha_{12})$. The representative curve of the function g splits the plane (α_{12}, e) into two parts. The valid set of e and α_{12} is the upper part as shown on the Figure 3. first valid domain :

$$\mathcal{D}_1 = \left\{ (\alpha_{12}, e) \in [\frac{\cos\theta}{2}, +\infty[\times[g(\alpha_{12}), 1] \right\} \qquad (30)$$

which depends only on α_{12} see figure (3). When $\theta > \frac{\pi}{2}$ the relative normal velocities are always positive irrespective of α_{12} and e.

We define also the function:

$$\begin{aligned} h: \ \mathbb{R} \ &\longmapsto \quad [0,1] \\ \alpha_{12} &\longmapsto \frac{2\alpha_{12}^2 - 2\cos(\theta) + 1}{2(\alpha_{12}^2 - \cos(\theta)\alpha_{12} + 1)} \end{aligned} \tag{31}$$

The valid set of e and α_{12} that ensures the positiveness of Δ is called $\mathcal{D}_2$ defined by :

$$\mathcal{D}_2 = \{(\alpha_{12}, e) \in [0, +\infty[\times [g(\alpha), 1]\} \tag{32}$$

The final valid set for e and α that complies with the unilateral constraints and the positiveness of Δ is the set $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$ which is depicted in the Figure (3). Since $\forall \alpha_{12} \in [\frac{\cos\theta}{2}, +\infty[$ we have $h(\alpha_{12}) \geq g(\alpha_{12})$ only the funtion g is represented on the Figure 3.The minimum of the function g is attained when $\alpha = \frac{\cos\theta}{2}$ which maps to $e = \dfrac{2 - \cos^2\theta}{4 - \cos^2\theta}$.

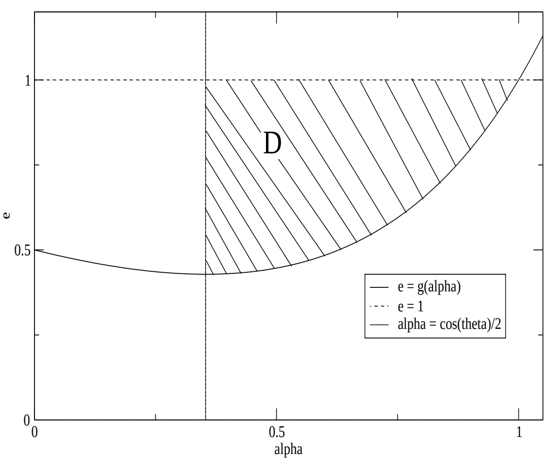

Figure 3. Valid domain of e and α for $\theta = \frac{\pi}{4}$.

Influence of the angle θ Let us consider different case studies on the angle θ.

Case 1. $\theta \in [0, \frac{\pi}{2}[$**.** For every e and α_{12} which respect the foregoing conditions, the solution is unique, respect the equation of motion, the unilateral constraints and finally, is energetically consistent. It's noteworthy that the solution of the monodimensional 3-ball chain is found again when we set $\theta = 0$.

Case 2. $\theta = \frac{\pi}{2}$**.** For $e = 1$, we have the following solution for the post-impact velocity :

$$\begin{cases} \dot{x_1}^+ = \dfrac{1}{2} \dfrac{-\alpha_{12} + 2\,\alpha_{12}{}^2}{-\alpha_{12} + 2\,\alpha_{12}{}^2 + 2} \\ \dot{y_1}^+ = \dfrac{1}{2} \dfrac{\sqrt{3}\left(-\alpha_{12} + 2\,\alpha_{12}{}^2\right)}{-\alpha_{12} + 2\,\alpha_{12}{}^2 + 2} \\ \dot{x_2}^+ = \dfrac{-1 + 2\,\alpha_{12}}{-\alpha_{12} + 2\,\alpha_{12}{}^2 + 2} \\ \dot{y_2}^+ = \dfrac{\sqrt{3}}{-\alpha_{12} + 2\,\alpha_{12}{}^2 + 2} \\ \dot{x_3}^+ = \dfrac{2\alpha_{12}}{-\alpha_{12} + 2\,\alpha_{12}{}^2 + 2} \\ \dot{y_3}^+ = 0 \end{cases} \tag{33}$$

and for the impulse :

$$\begin{cases} p_1 = \dfrac{1 + \sqrt{\Delta}}{2\,\alpha_{12}{}^2 + 2} \\ p_2 = \dfrac{(1 + \sqrt{\Delta})\alpha_{12}}{2\,\alpha_{12}{}^2 + 2} \end{cases} \tag{34}$$

In this particular position, the ball 3 must remain stationary. Hence, we have to enforce to $\alpha_{12} = 0$ in this case.

Case 3. $\theta \in]\frac{\pi}{2}, \frac{2\pi}{3}[$**.** In this case, the ball 3 must remain stationary. Hence, we have also to enforce to $\alpha_{12} = 0$ in this case.

Case 4. $\theta = \frac{2\pi}{3}$**.** This case is treated in the next section in a more general setting.

Preliminary conclusion To sum up we can say that our model is well posed for $\theta \in [0, \frac{2\pi}{3}[$ and $\tau = 0$ if we set $\alpha_{12} = 0$ for all $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}[$. This result is naturally correlated with the kinetic angle between the constraints. We will come back on this fact in § 5.

4.2 The influence of τ

The angle τ defines the orientation of the pre-impact velocity. We can state obviously that for $\tau \geq \frac{\pi}{2}$ only simple impact occurs. This fact is confirmed in the following results.

Case 1. $\theta \in [0, \frac{\pi}{2}[$ The obtained solution is unique, energetically consistent and reveals a dependance on the sign of $\cos\tau$. For instance, in the case, $\theta = 0$ and $e = 1$, the solution computed from this pre-impact data is:

$$
\begin{cases}
\dot{x_1}^+ = \dfrac{-\cos\tau\left(-1+2\,\alpha_{12}-2\,{\alpha_{12}}^2+\epsilon\right)}{2(1-\alpha_{12}+{\alpha_{12}}^2)}\\
\dot{y_1}^+ = -\sin\tau\\
\dot{x_2}^+ = \dfrac{\cos\tau(1+\epsilon)(\alpha_{12}+1)}{2(1-\alpha_{12}+{\alpha_{12}}^2)}\\
\dot{y_2}^+ = 0\\
\dot{x_3}^+ = \dfrac{\alpha_{12}\,\cos\tau\,(1+\epsilon)}{2(1-\alpha_{12}+{\alpha_{12}}^2)}\\
\dot{y_3}^+ = 0
\end{cases}
\tag{35}
$$

where $\epsilon = \text{sign}(\cos\tau)$ is 1 when $\tau \le \frac{\pi}{2}$ and -1 otherwise. The presence of the sign function allow us to treat in the same framework the case with no impact ($\tau \ge \frac{\pi}{2}$).

Case 2. $\theta = \frac{\pi}{2}$ When $\theta \ge \frac{\pi}{2}$ we face either single or no impact cases. Let us have a look at the solution of $\theta = \frac{\pi}{2}$. The solution still takes into account the no impact case through the function sign$(\cos\tau)$. This solution reads :

$$
\begin{cases}
\dot{x_1}^+ = \sin\tau\\
\dot{y_1}^+ = \dfrac{\cos\tau(1+2\,{\alpha_{12}}^2-\epsilon)}{2(1+{\alpha_{12}}^2)}\\
\dot{x_2}^+ = \dfrac{-\alpha_{12}\,\cos\tau(1+\epsilon)}{2(1+{\alpha_{12}}^2)}\\
\dot{y_2}^+ = \dfrac{\cos\tau(1+\epsilon)}{2(1+{\alpha_{12}}^2)}\\
\dot{x_3}^+ = \frac{\alpha_{12}\,\cos\tau(1+\epsilon)}{2(1+{\alpha_{12}}^2)}\\
\dot{y_3}^+ = 0
\end{cases}
\tag{36}
$$

As stated earlier in the preliminary conclusion of the subsection (4.1.1) , the case of $\theta \ge \frac{\pi}{2}$ enforces always the condition $\alpha_{12} = 0$ irrespective of the angle τ. Unless the $\alpha_{12} = 0$ the solution won't be intuitively plausible.

Case 2. $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}[$ Enforcing for all $\theta \ge \frac{\pi}{2}$ the impulse ratio $\alpha_{12} = 0$ yields always plausible results which are single impact.

So far we have been considering the two contacts cases providing us with 5 unknown in terms of 3 dynamical equations. Let's move on to the peculiar case of 3 contacts occasioned by $\theta = \frac{2\pi}{3}$.

Case 3. $\theta = \frac{2\pi}{3}$

Case 3.a $\tau \in [-\frac{\pi}{3}, 0]$
In this case we have one contact before the impact. Since the ball comes in contact with the two others, we have three impulses to take into account. The impulse ratio α_{13} can be now non zero as the impulse p_3 does not always vanish. For the particular value $\tau = -\frac{\pi}{6}$, we have the configuration shown on Figure 4. For the simplicity sake, we can say

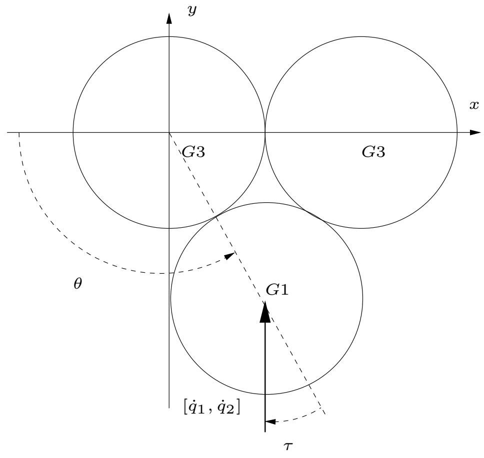

Figure 4. Bernoulli's problem

that $\alpha_{13} = \alpha_{12}$. The post-impact velocity is still very long to write. We have tried different values of the angle τ within this range. Below is the result we've got for $\tau = -\frac{\pi}{6}$:

$$
\begin{cases}
\dot{x_1}^+ = \dfrac{-\sqrt{3}(2\alpha_{12}^2+\alpha_{12}+1)}{4\alpha_{12}^2+\alpha_{12}+4}\\
\dot{y_1}^+ = \dfrac{2\alpha_{12}^2-\alpha_{12}-1}{4\alpha_{12}^2+\alpha_{12}+4}\\
\dot{x_2}^+ = \dfrac{-\sqrt{3}(1+2\alpha_{12})}{4\alpha_{12}^2+\alpha_{12}+4}\\
\dot{y_2}^+ = \dfrac{3}{4\alpha_{12}^2+\alpha_{12}+4}\\
\dot{x_3}^+ = \dfrac{5\alpha_{12}\sqrt{3}}{4\alpha_{12}^2+\alpha_{12}+4}\\
\dot{y_3}^+ = \dfrac{3\alpha_{12}}{2(4\alpha_{12}^2+\alpha_{12}+4)}.
\end{cases}
\tag{37}
$$

The model always yields consistent result for all $\tau \in [-\frac{\pi}{3}, 0]$.
Case 3.b $\tau \in [0, \frac{\pi}{2}]$
We see through calculation that for all $\tau \in [0, \frac{\pi}{2}]$ the provided solution is consistent. The sign function are present in this case also to help model the no impact or single impact cases.
Case 3.c $\tau \in [\frac{\pi}{2}, \frac{2\pi}{3}]$
In this range of τ the post-impact dynamics is always consistent irrespective of α's.

Preliminary conclusion The study of the angle τ allows us to say that the impulse ratio model work properperly when we set the impulse ratio to proper value according to the angle θ. The sign function help in modeling both the impact and no impact cases.

5 Towards a multiple impact law for general Lagrangian systems

Actually the aim of our study is to build a general impact law for m contacts in this frame:

$$\begin{cases} M\Delta.\dot{q} = P \\ v = H.\dot{q} \\ P = H^T.p \\ \Delta v \geq 0 \qquad p \geq 0 \\ \frac{1}{2}\dot{q}^{+T}.M.q^{+} = e.\frac{1}{2}\dot{q}^{-T}.M.\dot{q}^{-} \\ \mathcal{F}(p_j - \alpha_{ij}p_i, v, \theta_{ij}) \quad \text{is true} \end{cases} \tag{38}$$

where θ_{ij} is the kinetic angle between the gradient of the constraint h_i and h_j. The normal direction to a constraint $h_i(q) = 0$ is defined by (Brogliato, 1999):

$$n_i = \frac{M^{-1}(q)\nabla_q h_i(q)}{\sqrt{\nabla_q h_i(q)^T M(q)^{-1}\nabla_q h_i(q)}} \tag{39}$$

The kinetic angle between two constraints $h_i(q)$ and $h_j(q)$ is the angle θ_{ij} defined by:

$$\cos\theta_{ij} = n_i n_j. \tag{40}$$

The mapping $\mathcal{F}$ is some condition which have to be fulfilled by θ_{ij} and v in order to apply an impulse ratio between two unilateral constraints.

Clearly, the first criteria which has to be fullfilled is on the relative velocity v_i before the impact which need to belong to the opposite to the tangent cone to a constraint $h_i(q)$:

$$-v_i \in \mathcal{T}_i(q) = \{p| -\nabla h_i(q).p \geq 0\}. \tag{41}$$

If this condition is not fullfilled, we have no impact and on the contrary, we have a simple or a multiple impacts.

A second condition may be stated on the kinetic angle, more precisely, on the sequence of the kinetic angle between two adjoint contacts. A condition to have a transmission of the impulse relies on the values of this kinetic angles. We will try in the next section to be more precise coming back on the example of the pool and of a chain of n balls in the plane.

5.1 Back the pool example

The dependence of $\mathcal{F}$ on v In the case $\theta \in [0, \frac{2\pi}{3}[$, the angle τ gives a condition on the relative velocity v_1 belongs to the cone $-\mathcal{T}_1(q)$. If $\tau \geq \frac{\pi}{2}$, then $v_1 \in \mathcal{T}_1(q)$ and there is neither mutiple impacts nor simple impact. In the case of Bernoulli's problem, this is the same but with $v_1 \in -\mathcal{T}_1(q)$ and $v_1 \in -\mathcal{T}_3(q)$.

The dependence of $\mathcal{F}$ on θ_{ij} We evaluate the kinetic angle between the three constraints in our pool configuration. We found that:

$$n_1 n_2 = -\frac{1}{2}\cos\theta = \cos(\theta_{12}) \tag{42}$$

This means that the kinetic angle is directly related to the angle θ. Moreover the impulse ratio is also a function of the kinetic angle. We can venture to set up the function $\mathcal{F}$ we refered to in the system (38). Before setting the impulse ratio, one has to check if the two impulses are truly related. When we apply the foregoing preliminary conclusion in the section 4.1.1 we can say that two unilateral constraints may be coupled if

$$n_i.n_j < 0. \tag{43}$$

The underlying idea is that if the condition in (43) is not satisfied, then the local impulses p_i and p_j are not related. That is to say the impulse p_i has no influence in the jump in the normal relative velocity at the contact j and vice versa. So $\alpha_{ij} = \frac{p_j}{p_i}$ does not exist or has to be set to zero. Coming down to the present configuration, we have for instance to enforce $\alpha_{12} = 0$ in the section (4.1.1) when $\cos\theta \leq 0$. Naturally, we need to add some other conditions on the proximity of the contact in a kinematic chain.

5.2 The chain of n-balls in the plane

Let us consider a open chain of n-balls in the plane. the ball are numbered by an index i. We assume that the first ball possesses a velocity $[\dot{q}_1, \dot{q}_2]$ the others are stationary. We denote by $h_i(q), i \in 1 \ldots n-1$ the unilateral constraint between the ball i and the ball $i+1$. The kinematic angle between the constraint $h_i(q)$ and $h_{i+1}(q)$ is denoted by $\theta_{i,i+1}$. We can sum up the aforementioned condition by :

a) To have an impact (simple or multiple), the velocity v_1 must belong to the opposite of the tangent cone $\mathcal{T}_1(q)$:

$$v_1 \in -\mathcal{T}_1(q) \tag{44}$$

b) to have a multiple impact of degree k, the $k-1$ first kinetic angle $\theta_{i,i+1}$ must be greater than $\frac{\pi}{2}$:

$$\theta_{i,i+1} > \frac{\pi}{2}, \text{ i.e } n_i.n_{i+1} < 0, i = 1 \ldots k-1 \tag{45}$$

6 Conclusion and Perspectives

In this paper, we formulate more generally an impact law for rigid bodies in the case of Lagrangian systems. We can see that extending the monodimensional case of

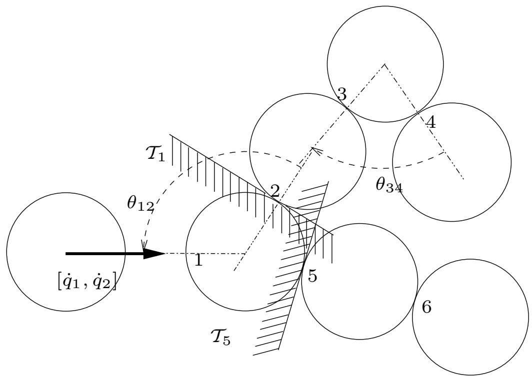

Figure 5. Transmission of impulses in an open chain of hard balls

the N-ball linear chain problem to the two-dimensional problem is somehow challenging as we still need more information about the impulse ratio and the angle θ. Our solution method works properly for $\theta \leq \frac{\pi}{2}$. The method provides us with a unique and energetically consistent solution. The solution also matches very well the experimental observation which we compute from Hertzian contact model. Still, the demand for a more precise definition of the impulse ratio arises. In this definition, the kinetic angle effect has to be taken into account. We sense through our simulation the dependence of α on the kinetic angle between the gradient of two constraints in the kinetic metric. A deeper study on the orthogonality of constraint may give some hint about coupled or decoupled impulses. The relation between the impulse ratio and the kinetic angle can be of particular interest. The study of the compliant system has shown the physical consistence of the impulse ratio.

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