Basics on numerical algorithms for Non Smooth Dynamical Systems

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Tuturial Lecture

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Outline

Scope

- Only Initial Value Problems (IVP).

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- Two typical examples of Non Smooth Dynamical Systems (NSDS) :
  - Linear Complementarity Systems
  - Lagrangian Dynamical Systems with contact and friction

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- Two typical examples of Non Smooth Dynamical Systems (NSDS) :
  - Linear Complementarity Systems
  - Lagrangian Dynamical Systems with contact and friction
- Two major kinds of time integration scheme :
  - Event–driven scheme. (the time–steps depend on the events)
  - Time–stepping scheme (the time–step does not depend on the events)

Linear Complementarity systems

- The Linear Complementarity System (LCS) may be defined by

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ 0 \leq y \perp \lambda \geq 0 \end{cases} \tag{1}$$

with $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}, D \in \mathbb{R}^{m \times m}$, for m constraints.
In the sequel, we consider the scalar case ($m = 1$)

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- Notion of Relative degree $r_{y\lambda}$
  Defining the Markov Parameters as

$$(D, CB, CAB, CA^2B, \ldots)$$

  the relative degree is the rank of the first non zero Markov Parameter.
  "The number of differentiation of y to obtain explicitly y in function of λ."

Linear Complementarity systems (Continued ...)

- Relative degree $r_{y\lambda} = 0$, D > 0, Trivial case
  - The multiplier $\lambda = \max(0, -D^{-1}Cx)$ is a Lipschitz continuous function of x
  - The numerical integration may be performed with any standard ODE solvers.

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- Relative degree $r_{y\lambda} = 1$, D=0, CB > 0
  - The multiplier λ is a function of time t, not necessarily continuous, for instance, of bounded variations (BV).
  - The numerical integration have to be performed with specific solvers (Event–Driven or Moreau's Time–stepping)

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- Higher Relative degree
  - The multiplier λ is a distribution of order $r_{y\lambda} - 1$.
  - Dedicated time-stepping integrators

Linear Lagrangian systems with Contact and Friction

- Lagrangian dynamical system :

$$M\ddot{q} + C\dot{q} + Kq = F_{ext}(t) + r \tag{2}$$

  - $q \in \mathbb{R}^n$: generalized coordinates vector.
  - $M \in \mathbb{R}^{n\times n}$: the inertia matrix
  - $K \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{n\times n}$: the stiffness and damping matrices,
  - $F_{ext}(t) : \mathbb{R} \mapsto \mathbb{R}^n$: given external force,
  - $r \in \mathbb{R}^n$ is the force due the nonsmooth law.

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- Linear relations.
  - Kinematical laws from the generalized coordinates to the local coordinates at contact.

$$y = H^T q + b, \dot{y} = H^T \dot{q}$$

    Mapping H : change of frame
  - By duality,

$$r = H\lambda$$

Linear Lagrangian systems with Contact and Friction

- Local frame at contact : $(\boldsymbol{n}, \boldsymbol{t})$

$$y = y_{\boldsymbol{n}}\boldsymbol{n} + y_{\boldsymbol{t}}, \quad \dot{y} = \dot{y}_{\boldsymbol{n}}\boldsymbol{n} + \dot{y}_{\boldsymbol{t}}$$

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$$0 \leq y_{\boldsymbol{n}} \perp \lambda_{\boldsymbol{n}} \geq 0 \quad \Longleftrightarrow \quad -\lambda_{\boldsymbol{n}} \in \partial\Phi_{\mathbb{R}^+}(y_{\boldsymbol{n}})$$

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- Coulomb's Friction, μ Coefficient of friction

$$\begin{cases} \dot{y}_{\boldsymbol{t}} = 0, \|\lambda_{\boldsymbol{t}}\| \leq \mu\lambda_{\boldsymbol{n}} \\ \dot{y}_{\boldsymbol{t}} \neq 0, \lambda_{\boldsymbol{t}} = -\mu\lambda_{\boldsymbol{n}}\text{sign}(\dot{y}_{\boldsymbol{t}}) \end{cases}$$

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- (Newton) Impact law, if necessary, e coefficient of restitution

$$\dot{y}_{\boldsymbol{n}}(t^+) = -e\dot{y}_{\boldsymbol{n}}(t^-)$$

Outline

Principle

- For a set of unilateral constraints :

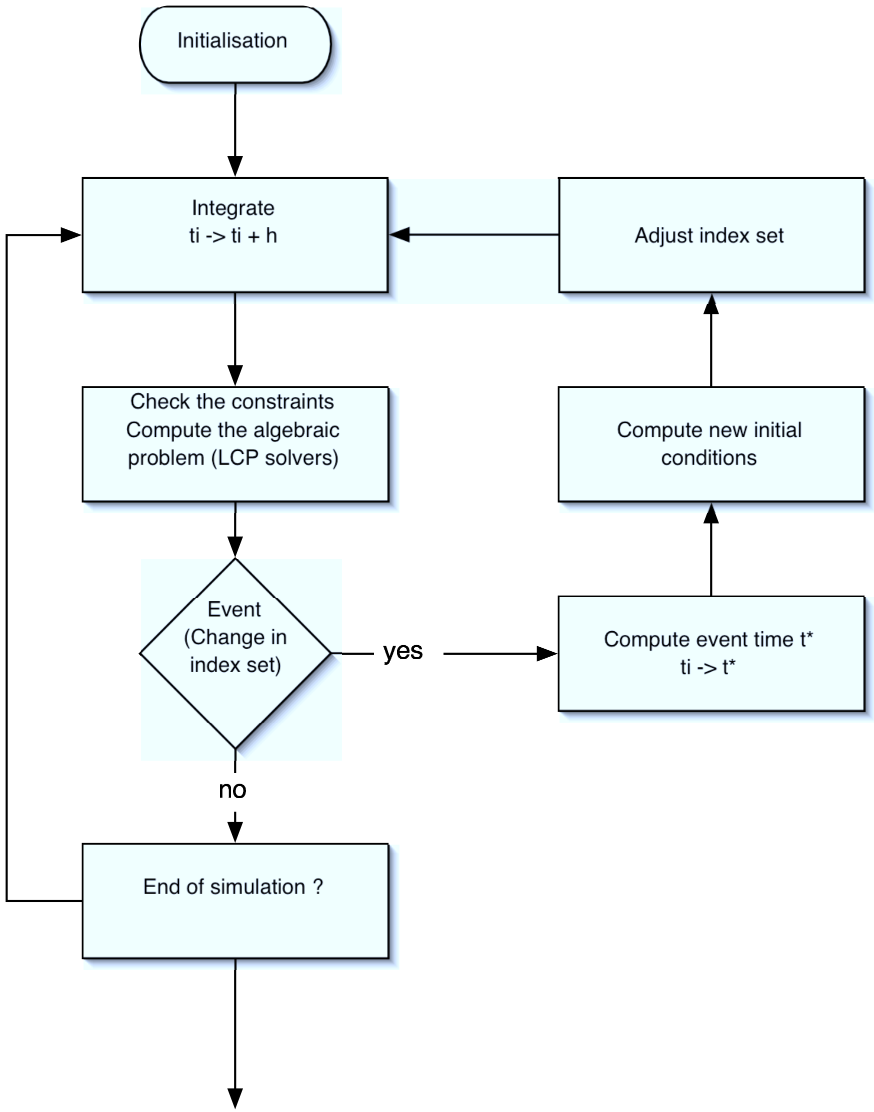
$$y_\alpha = h_\alpha(x) \geq 0, \alpha = 1 \dots \nu$$

  we define the index set of active constraints as :

$$I = \{\alpha, y_\alpha = 0\}$$

- Event = change in the index set of active constraints
- Stages in the time integration scheme:
  - With the assumption that there is no event in the time interval, (unilateral = bilateral), a standard time integration is done with any standard ODE solver.
  - At the end of the time step, one check the constraints with a relevant algorithm (e.g LCP solvers to avoid Delassus problem)
  - If the constraints are not satisfied, the switching time is found by an interpolation and a root finding procedure. At this switching time, both initial conditions and index set are updated (e.g. LCP solvers at various levels).

Pseudo-Algorithm



Comments

- For NSDS with relative degree ≥ 2, you need to solve an LCP problem in terms of the higher derivative of y.
  For instance, for Lagrangian systems, the unilateral constraints on displacement must be expressed in terms of the acceleration.
- The ODE integration solver must include a relevant treatment of bilateral constraints (DAE solvers) and an accurate root finding procedure.

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  - relative degree 0 or 1: ODE with possibly not continuous RHS,
  - relative degree 2: Measure differential equation (Lagrangian dynamical systems)
  - Higher Order : Higher Order Sweeping Process

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- The unknowns are chosen such that only real finite values are approximate:
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- The constraints are derived with respect to the time and treated at various levels to ensure the numerical stability.

Reformulation of the Dynamics

- Lagrangian dynamical system as a measure differential equation.

$$M\,dv + (Kq(t) + Cv(t))\,dt = F_{ext}(t)\,dt + R$$

where

- dt is the Lebesgue measure on $\mathbb{R}$
- dv is the Stieltjes measure (Differential measure) associated with the right continuous function $v(t)$ of bounded variations, such that :

$$dv((a,b]) = \int_{(a,b]} dv = v(b^+) - v(a^+)$$

- R is a measure due to the non smooth law
- $q(t)$ is the absolutely continuous displacement given by :

$$q(t) = q(t_0) + \int_{t_0}^{t} v(s)\,ds$$

Reformulation of the unilateral constraints

- Reformulation of the unilateral constraints in terms of derivatives :

$$\text{If } y(t) = 0, \text{ then } 0 \leq \dot{y} \perp \lambda \geq 0 \tag{2}$$

which can be stated equivalently as

$$-\lambda \in \partial \Psi_{V(q)}(\dot{y})$$

where $V(q)$ is the tangent cone of $\mathbb{R}^+$ at q and Ψ the indicator function.

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- If λ is a measure, the inclusion is extended considering the Radon-Nykodym derivative

$$\lambda'(t) = \frac{d\lambda}{d\nu} \in \Psi_{V(q)}(\dot{y})$$

where $d\nu$ is a nonnegative measure and λ is absolutely continuous with respect to $d\nu$

Discretization of the Dynamics

- Given a subdivision of a time interval, $\{t_0, t_1, \ldots, t_i, \ldots, t_N\}$, we evaluate of the measure differential equation on a time interval $(t_i, t_{i+1}]$ of length h :

$$M dv((t_i, t_{i+1}]) = \int_{(t_i,t_{i+1}]} M\, dv = M(v(t_{i+1}^+) - v(t_i^+))$$

$$M(v(t_{i+1}^+) - v(t_i^+)) = -\int_{t_i}^{t_{i+1}} Kq(t) + Cv(t)\, dt + \int_{t_i}^{t_{i+1}} F_{ext}(t)\, dt + \int_{(t_i,t_{i+1}]} R$$

- Evaluation of the displacement

$$q(t_{i+1}) = q(t_i) + \int_{t_i}^{t_{i+1}} v(s)\, ds$$

Discretization of the Dynamics Continued

- The measure $R((t_i, t_{i+1}])$ of the time-interval $(t_i, t_{i+1}]$ is kept as primary unknown :

$$R_{i+1} = R((t_i, t_{i+1}])$$

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- Interpretation : The measure R may be decomposed as follows :

$$R = R_a\, dt + R_s$$

where $R_a\, dt$ is the abs. continuous part of the measure R and R_s the singular part.

  - Impulse : If $R_a = 0$ and $R_s = P\delta_{t_{i+1}}$ then $R_{i+1} = P$
  - Continuous multiplier : If $R_a(t) = f(t)$ and $R_s = 0$ then $R_{i+1} = \int_{t_i}^{t_{i+1}} f(t)\, dt$

Discretization of the Dynamics

- Notations :

$$v_i \approx v(t_i^+), \quad q_i \approx q(t_i)$$

- Approximation of the integral of functions : θ-method

$$\int_{t_i}^{t_{i+1}} Kq(t) + Cv(t)\,dt \approx h\left[\theta(Kq_{i+1} + Cv_{i+1}) + (1-\theta)(Kq_i + Cv_i)\right]$$

$$\int_{t_i}^{t_{i+1}} F_{ext}(t)\,dt \approx h\left[\theta F_{ext}(t_{i+1}) + (1-\theta)F_{ext}(t_i)\right]$$

- Evaluation of the displacement: θ-method

$$q_{i+1} = q_i + h\left[\theta v_{i+1} + (1-\theta)v_i\right]$$

Discretization of the Dynamics Continued

- Complete set of discrete equations:

$$\begin{cases} M(v_{i+1} - v_i) = h\left[\theta(Kq_{i+1} + Cv_{i+1}) + (1-\theta)(Kq_i + Cv_i)\right] \\ \qquad\qquad +h\left[\theta F_{ext}(t_{i+1}) + (1-\theta)(F_{ext}(t_i)\right] + R_{i+1} \\ q_{i+1} = q_i + h\left[\theta v_{i+1} + (1-\theta)v_i\right] \end{cases}$$

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- One step linear system :

$$v_{i+1} = v_{free} + hWR_{i+1}$$

with

$$W = \left[M + h\theta C + h^2\theta^2 K\right]^{-1}$$

$$v_{free} = v_i + W\left[-hCv_i - hKq_i - h^2\theta K v_i + h\left[\theta F_{ext}(t_{i+1}) + (1-\theta)F_{ext}(t_i)\right]\right]$$

Discretization of the constraints

- Discretization of the relations :

$$
\begin{aligned}
y_{i+1} &= H^T q_{i+1} + b \\
\dot{y}_{i+1} &= H^T v_{i+1} \\
R_{i+1} &= H \lambda_{i+1}
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- Discretization of an unilateral constraint :
  A natural way :

$$0 \leq y_{i+1} \perp \lambda_{i+1} \geq 0$$

  in terms of velocity

$$\text{If } y^p \leq 0, \text{ then } 0 \leq \dot{y}_{i+1} \perp \lambda_{i+1} \geq 0$$

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- Newton Impact law $\dot{y}^e_{i+1} = \dot{y}_{i+1} + e\dot{y}_i$

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Summary

One step linear problem

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➜ One step LCP in terms of $\dot{y}^e_{i+1}$ and λ_{i+1} :

$$\begin{aligned} \dot{y}^e_{i+1} &= H^T \dot{q}_{free} + hH^T WH\lambda_{i+1} + e\dot{y}_i \\ y^p &= y_i + \frac{h}{2}\dot{y}_i \\ \text{If } \quad y^p &\leq 0, \ \text{then } 0 \leq \dot{y}^e_{i+1} \perp \lambda_{i+1} \geq 0 \end{aligned}$$

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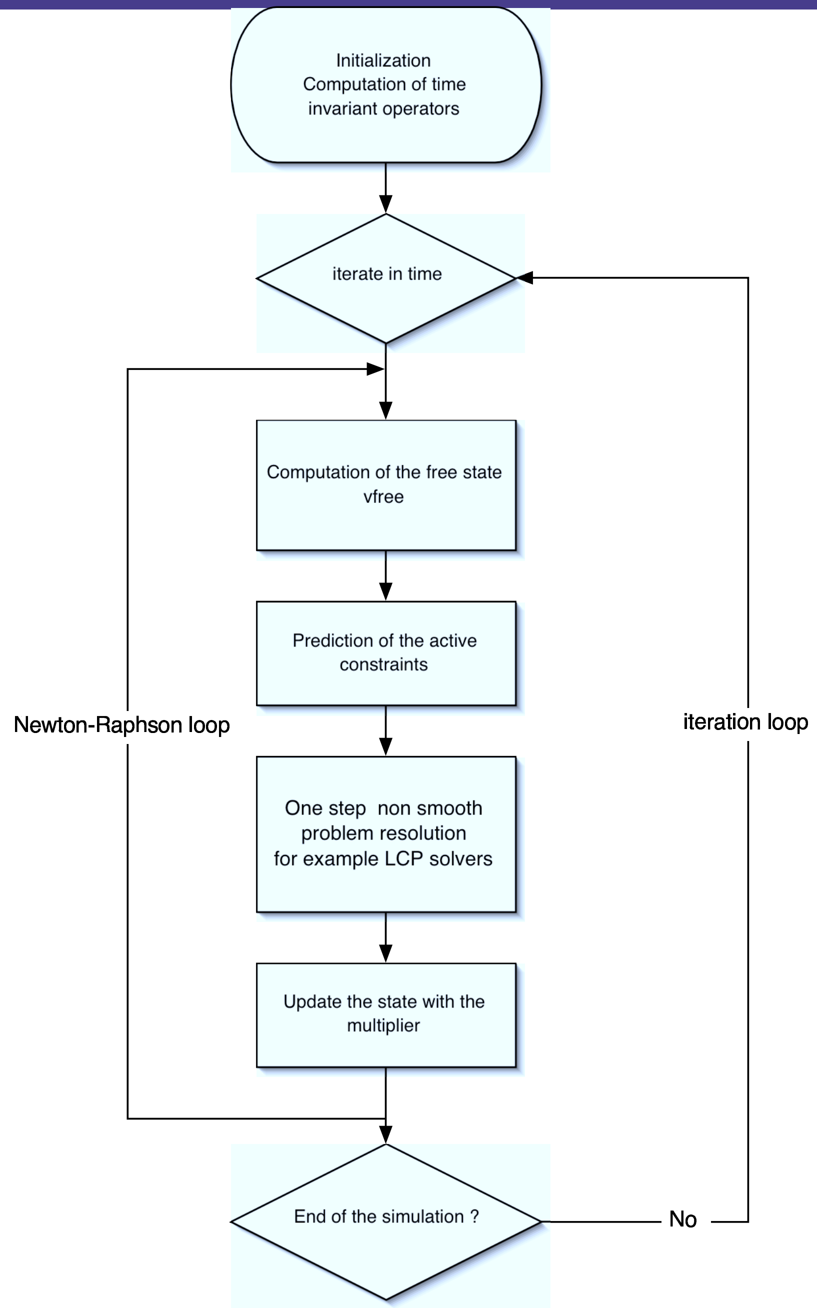

Initialization
Computation of time invariant operators
iterate in time
Computation of the free state vfree
Prediction of the active constraints
One step non smooth problem resolution for example LCP solvers
Update the state with the multiplier
End of the simulation ?
No
Newton-Raphson loop
iteration loop

Linear complementarity system

- Direct application of a Backward Euler Scheme :

$$\left\{\begin{array}{l} \dfrac{x_{k+1} - x_k}{h} = A x_{k+1} + B\lambda_{k+1} \\ y_{k+1} = C x_{k+1} + D\lambda_{k+1} \\ 0 \leq \lambda_{k+1} \perp y_{k+1} \geq 0 \end{array}\right.$$

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  ➜ Direct equivalence with the Moreau's Time-stepping scheme
- Relative degree 2
  ➜ inconsistency of the variable λ_{k+1} which tends toward $+\infty$ when $h \longrightarrow 0$

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  - Relative degree 2
    ➜ inconsistency of the variable λ_{k+1} which tends toward $+\infty$ when $h \longrightarrow 0$
- Moreau's Time–stepping scheme for a relative degree 2:
  - The primary unknown is $R_{i+1} = h\lambda_{k+1}$,
  - The unilateral constraint is set on $\dot{y}_{k+1}$

➜ See the illustration on the LCS

Outline

Event–Driven - Advantages and disadvantages

- Advantages :
  - Low cost implementation (re-use of existing ODE solvers).
  - Higher-order accuracy on free motion.
  - Pseudo-localisation of the time of events with finite time-step.

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- Weaknesses
  - Numerous events in short time.
  - Accumulation of impacts.
  - No convergence proof

Time–stepping - Advantages and disadvantages

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  - Convergence proofs (stability and consistency) ➜ Existence and uniqueness results
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Time–stepping vs. Event–Driven

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  - strong accuracy requirements on the free motion
  - sparse events
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  - strong accuracy requirements on the free motion
  - sparse events
  - low number of constraints
- Time–stepping schemes are suitable for simulations with :
  - dense events and accumulation
  - high number of constraints

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Consider the following LCS of relative degree 2:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \lambda \\ y = x_1 \\ 0 \leq y \perp \lambda \geq 0 \end{cases}$$

with inelastic reinitialization mapping (if $y(t) = 0, \quad \dot{y}(t^+) = 0$)

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- Initial condition $x(0^-) = (0, -1)^T$

  Backward Euler scheme: $x_k = (0,0), \forall k, \lambda_1 = \frac{1}{h}, \lambda_k = 0$

  Moreau's time stepping: $x_k = (0,0), \forall k, \lambda_1 = 1, \lambda_k = 0$

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  Moreau's time stepping: $x_k = (0,0), \forall k, \lambda_1 = 1, \lambda_k = 0$

- Initial condition $x(0^-) = (-1, -1)^T$

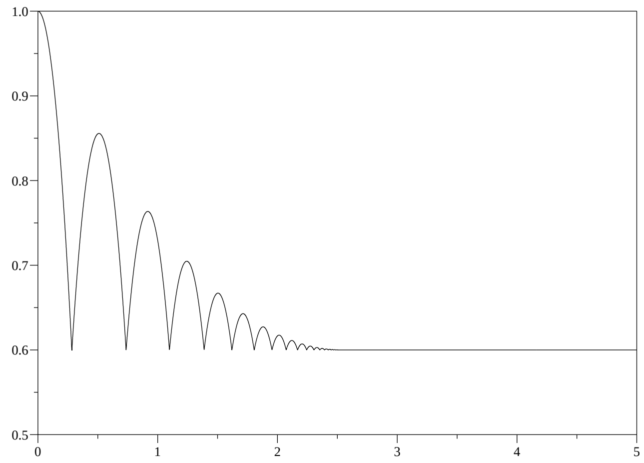
  Backward Euler scheme: $x_k = (k, \frac{1}{h}), \forall k, \lambda_1 = \dfrac{1}{h^2}, \lambda_k = 0$

  Moreau's time stepping: $x_k = (-1, 0), \forall k, \lambda_1 = 1, \lambda_k = 0$

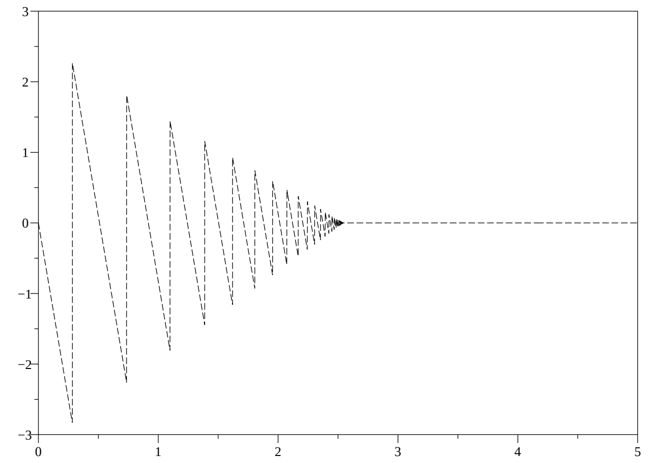
  Extended Moreau's time stepping: $x_k = (0,0), \forall k, \mu_1 = 1, \lambda_1 = 1, \lambda_k = 0, \ \mu_k = 0$

The Boucing ball example with time–stepping

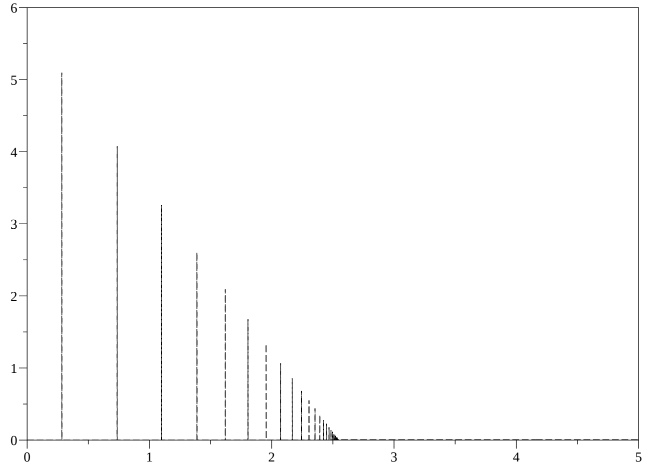
$$\begin{cases} m\ddot{q} = -mg + \lambda \\ y = q \\ 0 \leq y \perp \lambda \geq 0 \\ \text{if } y(t) = 0, \\ \quad \dot{y}(t^+) = -ey(t^-) \end{cases}$$



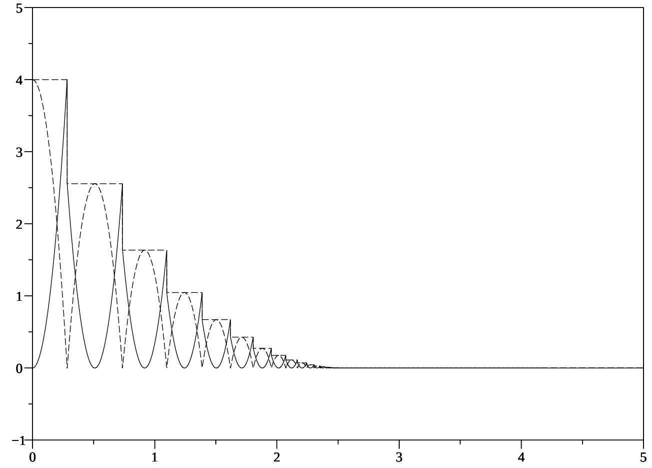
Position of the ball vs. Time



Velocity of the ball vs. Time



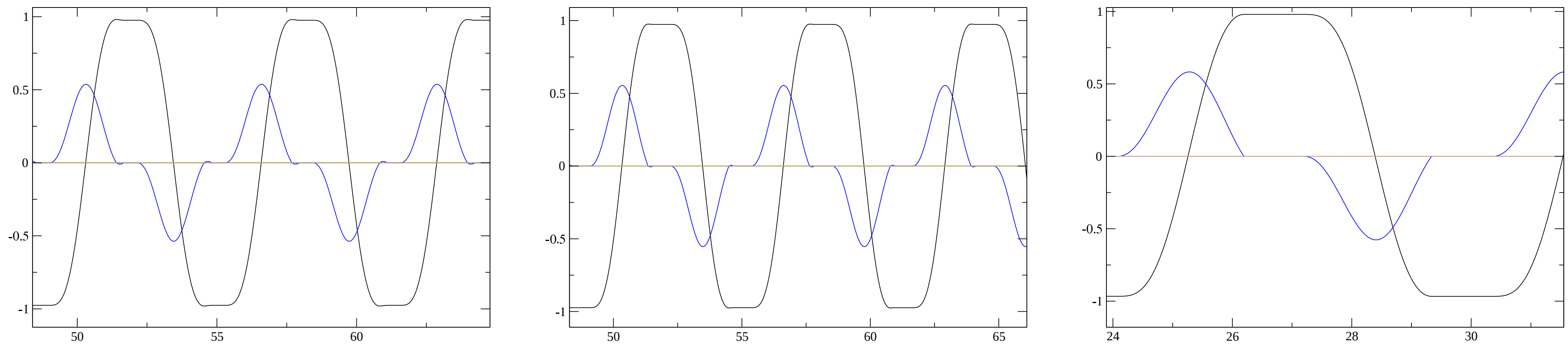
Reaction due to the contact force vs. Time



Energy balance vs.time

A friction oscillator

$$\begin{cases} \ddot{q} + q = sin(\omega t) + r \\ y = q, r = \lambda \\ \begin{cases} \dot{y} = 0, \|\lambda\| \leq \mu \\ \dot{y} \neq 0, \lambda_t = -\mu \text{sign}(\dot{y}) \end{cases} \end{cases}$$



$\mu = 0.025$ $\mu = 0.05$ $\mu = 0.1$

Position and velocity of the oscillator vs. Time

Outline

Further reading:

- Event–Driven
  - F. Pfeiffer & C. Glocker. Multibody Dynamics with Unilateral Contact, John Wiley & Sons, 1996
  - M. Abadie, Dynamic Simulation of Rigid bodies: Modelling of Frictional contact, Impact in Mechanical Systems, analysis and modelling, B. Brogliato ed., LNP 551 Springer Verlag
- Time-stepping
  - J.J. Moreau, Evolution Problem Associated with a Moving Convex Set in a Hilbert Space, Journal of Differential Equations, pp 347-374 1977
  - J.J. Moreau, Unilateral contact and dry friction in finite freedom dynamics, CISM 302, Springer Verlag, pp 1-82, 1988
  - J.J. Moreau, Some numerical methods in multibody dynamics: Application to granular materials, European Journal of Machanics-A/Solids, pp 93-114, 1994.