NUMERICAL MODELLING OF THREE DIMENSIONAL DIVIDED STRUCTURES BY THE NON SMOOTH CONTACT DYNAMICS METHOD APPLICATION TO MASONRY BUILDINGS

V. Acary*,†, M. Jean*
***Laboratoire de Mécanique et d'Acoustique de Marseille**
IMT / ESM2 Technopole de Chateau Gombert
13451 Marseille Cdx 20, France
†GAMSAU/ MAP
Ecole d'architecture de Marseille 184, avenue de Luminy
13288 Marseille cdx 9, France

Abstract

This paper outlines a computational method for modelling 3D divided structures by means of interface models, characterized by unilateral properties. The theoretical framework belongs to the field of non-smooth mechanics which aims at solving problems where severe time and space discontinuities are encountered. Multi-valued and stiff interfaces laws, e.g., Signorini's condition and Coulomb's friction, are solved using tools and formalisms provided by convex analysis. This general framework is adapted to micro-modelling approach of masonry structures, specifying interfaces models to mortar joints behaviour. The various stages in the development and implementation of an algorithm are delineated. Reaching a quasi-static equilibrium of floating structure is discussed and some numerical applications are presented on didactic tests.

1 Introduction

1.1 Mechanics of masonry

Masonry structures play an important part in ancient and contemporary civil engineering buildings. Regardless of the nature of masonry, brick, stone or concrete masonry, bonded with mortar joints or not, masonry appears as a divided medium made of different components with quite different mechanical behaviors. This kind of mechanical system may be seen at two different scales: at the macro-mechanics scale, it may be assumed as some homogenized material; at the micro-mechanics scale, every component is explicitly described. Strong material heterogeneity involves localized stress-strain response to loading. Cracks, localization of deformation, are the major features of this response. The mechanical behavior of the whole structure, resulting from non-linear effects of each component, appears itself to be strongly non-linear and cannot easily be described by analytical ways. Traditional limit analysis theory [11, 21] is still the most accurate analytical method to get limit loads, but generally, post-fracture behaviors, structural effects may hardly be taken into account without strong assumptions like, in-plane behavior or simplified failure surface assumptions [4]. Thus, numerical methods are commonly admitted as a tool to explore structural and local behavior.

It still remains the more controversial question of adopting divided or continuum mechanics models. Traditional mechanics of continuum media come up against difficulties when failure and large deformations have to be taken into account. For instance, homogenization techniques provide reliable and mathematically proved [24] equivalent laws only in the case of elasto-plasticity with hardening. Under the assumptions of plasticity with softening or damage behavior, or in general when non convex energy functional behaviors are encountered, numerical modelling suffers from a lack of convergence of the weak problem, exhibited, for instance, by mesh sensitivity [19]. Higher orders gradients theory permit localization of deformation without losing ellipticity for quasi-static problems and hyperbolicity for dynamics problems [9, 10] but introduce some lack of definition in boundary conditions.

1.2 The proposed approach

The Non Smooth Contact Dynamic Method (shortly NSCD) initiated by J.J. Moreau [16] and M. Jean [13] during the last decade is presented in this paper. This method is applied to masonry as a collection of deformable bodies bonded with unilateral constraints up to modelize the failure of joints. Modeling take advantages of specificities of failure mechanism of masonry, generally located in joints, introducing damage behavior through interfaces laws. Even if this approach is more debatable for diffuse damage, this approach seems to be relevant for crack propagation [18, 12] and therefore natural for masonry structures. Behavior of deformable bodies inherit from continuum elastic behavior laws. The medium is then considered as a discrete medium by the use of more or less complicated constraints between blocks, here after discussed.

Numerical way to deal with such discrete medium are numerous. Algorithms dedicated to multi-body systems [22], distinct element method (DEM) [8], discontinuous deformation analysis (DDA) [23] are examples of such methods with various ways to catch the discrete character of the media. The original aspect of the NSCD method with respect to [8], is given by the following major features :

- finite element method is used to represent continuum elastic law for each body,
- an implicit scheme is used for the dynamical equations which may be degenerate into quasi-static incremental evolution,
- a non-smooth treatment is applied to stiff frictional contact laws (non regularized),

- at each step, constraints are solved using a non-linear block Gauss-Seidel algorithm.

Different contact laws may be chosen according to experimental data [5]. For instance, in the case of bonding of dry stones, an appropriate model is the unilateral Signorini condition and the Coulomb dry friction law. When ancient mortar, composed of sand and quick-lime, is considered, an interface debonding model is used together with a more or less complicated damage law chosen with respect to some characteristic scale of loading. In quasi-static evolution, the Mohr-Coulomb failure criterion is usually assumed for mortar. A more sophisticated kind of model involves a progressive surface damage evolution driven by energetic criteria and allows a reliable and accurate description of the local evolution of mechanical fields: traction and shear stresses, displacement jump across the lips of the cracks [15, 6, 20]. In dynamic evolution, for instance seismic excitation, micro-mechanically consistent binary representation of cohesive law [12] may be proposed. This law may be seen as the limit case of the damageable cohesive zone model, providing brittle fracture evolutions. The average behavior of the structure together with structural effects is a classical failure behavior. Introducing these different cohesive frictional laws results in minor changes in the NSCD algorithms.

In this paper, the Non Smooth Contact Dynamics method and theoretical frame of interfaces laws is briefly presented. One discusses how a quasi-static equilibrium is obtained for a constrained multi-body systems. For a complete comprehensive treatment of such methods, we refer the reader to [13, 16, 2]. Choice of interfaces laws according to experimental studies is enlightened and particularly the choice of materials parameters. Some results will be given on masonry buildings submitted to various loads: for instance, 3D walls and vaults undergoing settlements of grounds and 3D dome. Results will be discussed according to the choice of interface laws, the choice of parameters driving the numerical algorithm (time or load increments, accuracy)

2 Dynamics with unilateral constraints

2.1 Motion equation

A standard space discretization, provided for instance, by a variational approximation, is made and yields to a well-known equation which sums up the dynamical behavior of a mechanical system:

$$\boldsymbol{M}(\ddot{\boldsymbol{q}}, \boldsymbol{q}, t) + \boldsymbol{Q}(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) = \boldsymbol{P}(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) \tag{1}$$

$\boldsymbol{q}$ is then a discrete representative of the space continuous position function. If rigid bodies are considered, $\boldsymbol{q}$ may be the standard configuration variable like displacements and rotations. The inertial operator, $\boldsymbol{M}$, is assumed to be linear with respect to $\ddot{\boldsymbol{q}}$, i.e., $\boldsymbol{M}(\ddot{\boldsymbol{q}}, \boldsymbol{q}, t) = \boldsymbol{M}(\boldsymbol{q}, t)\ddot{\boldsymbol{q}}$. $\boldsymbol{Q}$ is the non-linear operator which represents the discrete internal forces and $\boldsymbol{P}$ the discrete external applied loads.

When unilateral constraints are considered, Equation (1) must be understood in a distributional sense. Indeed, velocity's discontinuities do not allow to define correctly acceleration as a standard derivative of a smooth function. An other equivalent way to deal with such differential equations is to introduce differential measures. In the NSCD method, the primary unknown is the velocity and more exactly, the right continuous local bounded variation function of time $\boldsymbol{u}(t)$ such that:

$$\boldsymbol{q}(t) = \boldsymbol{q}(t_0) + \int_{t_0}^{t} \boldsymbol{u}(\tau)\, d\tau$$

A Stieltjes measure $d\boldsymbol{u}$ may be associated with $\boldsymbol{u}(t)$ such that $\int_{t_1}^{t_2} d\boldsymbol{u} = \boldsymbol{u}(t_2^+) - \boldsymbol{u}(t_1^-)$. Equation (1) may be rewritten under measure differential equation:

$$\boldsymbol{M}(\boldsymbol{q}, t)d\boldsymbol{u} + \boldsymbol{Q}(\boldsymbol{u}, \boldsymbol{q}, t)dt = \boldsymbol{P}(\boldsymbol{u}, \boldsymbol{q}, t)dt$$

where dt is a traditional Lebesgue measure.

Furthermore, the impulsive force on dt , $\boldsymbol{P}dt$ may be split into two parts such that:

$$\boldsymbol{P}(\boldsymbol{u}, \boldsymbol{q}, t)dt = \boldsymbol{F}(\boldsymbol{u}, \boldsymbol{q}, t)dt + d\boldsymbol{R}$$

where $\boldsymbol{F}$ is the bounded and continuous force, and $d\boldsymbol{R}$ is the contact impulsion due to the respect of unilateral constraints which can be formulated as $d\boldsymbol{R} = \boldsymbol{R}_\nu d\nu$ with $d\nu$ is a non negative real measure. One can sum up dynamical evolution of a mechanical system by the following equation:

$$\boldsymbol{M}(\boldsymbol{q}, t)d\boldsymbol{u} + \boldsymbol{Q}(\boldsymbol{u}, \boldsymbol{q}, t)dt - \boldsymbol{F}(\boldsymbol{u}, \boldsymbol{q}, t)dt = \boldsymbol{R}_\nu d\nu \tag{2}$$

2.2 Unilateral constraints and interface laws

2.2.1 Local frame and gap function

Two important notions are introduced in this section in order to formulate basic contact-friction constraints and to specify conventions of notations. Let's consider two bodies $\overline{\Omega}^1, \overline{\Omega}^2$, for each candidate particle to contact, $\boldsymbol{q}_1 \in \partial\Omega^1$, a proximal particle of $\partial\Omega^2$, $\boldsymbol{q}_2$, is defined by performing the following minimum: $\min_{\boldsymbol{q}\in\partial\Omega^2} \{d(\boldsymbol{q}_1, \boldsymbol{q})\}$. $\boldsymbol{q}_2$ is naturally the normal projection of $\boldsymbol{q}_1$on $\partial\Omega^2$. $\{\boldsymbol{q}_1, \boldsymbol{q}_2\}$ is called the contact pair composed respectively by the candidate and the antagonist particles.

A orthonormal frame $(\boldsymbol{n}\,\boldsymbol{t}\,\boldsymbol{s})$ is introduced at the locus of contact. $\boldsymbol{q}_2$ is its origin and $\boldsymbol{n}$ is the unit outward normal vector to $\overline{\Omega}^2$ at $\boldsymbol{q}_2$, and $\boldsymbol{t}, \boldsymbol{s}$ complete the orthonormal frame, so-called local frame (see Figure).

The gap function g is given by $g(\boldsymbol{q}_1) = \overline{\boldsymbol{q}_2\boldsymbol{q}_1}$. This signed real function is defined by the orientation induced by $\boldsymbol{n}$ along the straight line $(\boldsymbol{q}_1, \boldsymbol{q}_2)$. Moreover, relative velocity $\boldsymbol{v}$ can be decomposed in normal and tangential components in the local frame : $\boldsymbol{v} = v_N\boldsymbol{n} + \boldsymbol{v}_T$. It's noteworthy that v_N is the time derivative of the gap function.

2.2.2 Signorini's Condition

A basic unilateral constraint is the so-called Signorini's condition, which ensures the impenetrability between two bodies. Following the definition of the gap and the chosen convention on signs, constraint of impenetrability can be formulate as : $g(\boldsymbol{q}1) \geq 0$. Reaction $\boldsymbol{r}$ which acts from the antagonist to the candidate particle, can be submitted to the same decomposition as velocity: $\boldsymbol{r} = r_N\boldsymbol{n} + \boldsymbol{r}_T$. The dual constraint of impenetrability on r_N implies $r_N \geq 0$.

A last equation governing unilateral contact between two bodies is added and complete the complementarity problem of Signorini:

$$g \geq 0 \qquad r_N \geq 0 \qquad g r_N = 0$$

This set of equation can be summarized under sub-differential inclusions:

$$-r_N \in \partial\Psi_{\mathbb{R}^+}(g) \quad g \in \partial\Psi^*_{\mathbb{R}^+}(-r_N) = \partial\Psi_{\mathbb{R}^-}(-r_N) \quad (3)$$

where $\Psi^*$is dual function of Ψ, result of the Legendre-Fenchel transformation.

This set of relation between the gap and normal reaction will be shortly denoted as

$$(g, r_N) \in GR - Signorini$$

This notation outlines the fact that the couple (g, r_N) holds in the graph of the multi-application which represents the Signorini Condition (see Figure). Moreover, inclusions (3) ensure that (g, r_N) follows the assumption of normal dissipation (normality condition). This property allows the use of upper and lower bounds of limit analysis theorem useful to carry out collapse mechanisms. Indeed, complementary equations governing unilateral constraints are similar to those which govern no-tension material. Major discrepancy is just found in the locus of failure which is a priori given by the geometry of interfaces corresponding to joints.

In previous section, velocity is presented as a primary variable and this fact is at the heart of our approach of dynamical submitted to unilateral constraints. As Moreau has established in [17], a equivalent form of Signorini's condition may be formulated in terms of relative velocity with some mathematical cares. Indeed, if inequality on g is verified at $t = 0$, and following inclusion $-v_N \in \partial\Psi_{\mathbb{R}^-}(-r_N)$ is holds $\forall t \in [0, T]$, equality on g is verified $\forall t \in [0, T]$. This non-trivial result allows to compute Signorini condition directly in terms of velocity:

$$-r_N \in \partial\Psi_{\mathbb{R}^+}(v_N) \qquad v_N \in \partial\Psi^*_{\mathbb{R}^+}(-r_N) = \partial\Psi_{\mathbb{R}^-}(-r_N)$$

2.2.3 Coulomb's friction

Coulomb's friction is introduced to modelize shear strength proportional to normal load. The use of tangent relative velocity is the natural way to formalize dry friction of Coulomb such that :

- stick phase $\|\boldsymbol{v}_T\| = 0 \qquad \|\boldsymbol{r}_T\| \leq k$
- slip phase $\|\boldsymbol{v}_T\| > 0 \qquad \|\boldsymbol{r}_T\| = k$ and $\{\dfrac{\boldsymbol{r}_T}{\|\boldsymbol{r}_T\|} = \dfrac{\boldsymbol{v}_T}{\|\boldsymbol{v}_T\|}\}$

The threshold k is defined for Coulomb's friction proportional to r_N by a coefficient of friction μ. i.e. $k = \mu r_N$. This set of relation may be summarized in the following generic term:

$$(\boldsymbol{v}_T, \boldsymbol{r}_T) \in GR - Coulomb(r_N)$$

As soon as the normal component is assumed an known value, these pair of statements lets itself be expressed as a law of resistance deriving from a pseudo-potential $\Psi_{C(\mu r_N)}$. Therefore, equivalent inclusion forms of Coulomb's friction is given by these formulas :

$$\boldsymbol{v}_T \in \partial\Psi_{C(\mu r_N)}(\boldsymbol{r}_T) \qquad \boldsymbol{r}_T \in \partial\Psi^*_{C(\mu r_N)}(\boldsymbol{v}_T)$$

where C is the disk of radius $k = \mu R_N$ i.e. $C(\mu r_N) = \{\boldsymbol{r}_T, \|\boldsymbol{r}_T\| \leq \mu r_N\}$.

A maximum dissipation principle may be also expressed to carry on an analogy with plasticity:

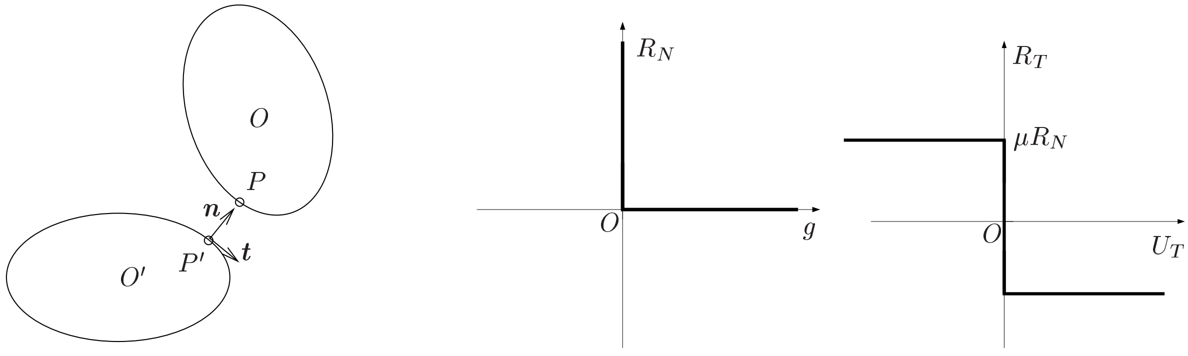
$$\boldsymbol{r}_T \in C(\mu r_N) \quad \forall \boldsymbol{s}_T \in C(\mu r_N) \quad (\boldsymbol{s}_t - \boldsymbol{r}_T)\boldsymbol{v}_{_T}$$

As we have remarked in the previous section, in the case of frictionless constraint, equations are equivalent to those formulated in no-tension material. Some analogies may be also made with plasticity theory but some special cares must be taken. With dry friction, extremely important difference between Coulomb friction and the apparently corresponding resistance to plasticity have to be outlined. Indeed, Coulomb friction do not follow the assumption of normality and this fact is partially the result of the dependence of $\Psi_{C(\mu r_N)}$ with the respect to r_N. Likewise, the absence of a genuine potential for this frictional contact problem reflects the non-associated character of the slip rule. Therefore no dilatancy of interfaces is induced by sliding rates. Limit analysis theorem have to be replaced by Radenkovic theorem for non-associated plasticity. Considering associated problems with constitutive models following normality assumption, such theorems provide only bounds which contains limit states. For instance, these bounding problems for dry friction are the zero friction problem and the totally glued problem.

2.2.4 More sophisticated interfaces laws

Signorini's condition and Coulomb's friction law are the basic constitutive equations for modelling the unilateral behaviour in direct tension and the strength threshold in shear loading for joints in masonry structure. If dry stone structures are considered, these laws seem well-suited to fit mechanical behavior of joints [5]. For joints which have decayed in time, any load leads to an irreversible damage which transforms mortar in a pulverulent state. In this case, cohesion of joints may be neglected and tensile strength is due to friction, generated by this third body between the bodies. In this case, dry friction Coulomb's law still remains a good issue to modelize shear strength with threshold.

On the other hand, in many masonry structure, cohesion, i.e., tensile strength plays an important part in the mechanical response of the structure. Likewise, standard sliding threshold of coulomb, which is a reliable parameter in large sliding when mortar are completely damaged by shear, can underestimate the shear strength when joints is still sound. Moreover, assumption of perfectly rigid interfaces in compression may be inappropriate for some structural analysis. Indeed, mortar can follow elastic behaviour in small perturbations. This elastic behaviour, coupled to cohesion may a important factor of accommodation to external loads and redistribution of stresses.



(a) Local frame (b) The Signorini graph (c) Coulomb graph

Figure 1: Basic interface laws

These various physical evidences are introduced through mere modification of Signorini's condition and Coulomb's friction. Our choice is to keep Signorini's condition and Coulomb's friction as fundamental constitutive models in order to build enhanced interface model. It is motivated by the fact that impenetrability and dry friction remain the most important characteristics of the mechanical response when joints are completely damaged. Therefore, more elaborated interface laws are written with the idea that these modifications must vanish after large perturbations and/or cycles of loading.

Formally, enrichment of method are introduced through auxiliary variables as follows:

$$\begin{aligned}\tilde{g} &= g - f(r_N)\\ \tilde{r}_N &= r_N - \rho_N([u_N], v_N, \alpha_i)\\ \tilde{\boldsymbol{r}}_T &= \boldsymbol{r}_T - \boldsymbol{\rho}_T([u_T], \boldsymbol{v}_T, \alpha_i)\end{aligned}$$

$[u_N], [u_T]$ are respectively the normal and tangent displacement across the interface. Its definitions are obvious when the problem is to deal with small displacements and then, small sliding. It is a more debated issue when the problem involves finite sliding. $(\alpha_i)_{i\in[1\ldots m]}$ are internal variables of an assumed material surface like cohesion, damage, etc ...

Let's consider $\Psi_N(g, r_N) \in \mathbb{R}^2 \longrightarrow (\tilde{g}, \tilde{r}_N) \in \mathbb{R}^2$ and $\Psi_T(\boldsymbol{r}_T) \in \mathbb{R}^2 \longrightarrow (\tilde{\boldsymbol{r}}_T) \in \mathbb{R}^2$ which perform the substitution of variables . Ψ_N, Ψ_T are chosen as bijection for a set of internal variables, $(\alpha_i)_{i\in[1\ldots m]}$, a priori known. This property allows the use of a single numerical solver for various interfaces laws. These auxiliary variables are assigned to follow respectively graph of the multiapplication such that:

$$\begin{aligned}(\tilde{g}, \tilde{r}_N) &\in GR - Signorini\\ (\boldsymbol{v}_T, \tilde{\boldsymbol{r}}_T) &\in GR - Coulomb(\tilde{r}_N)\end{aligned}$$

Functions $\rho_N, \boldsymbol{\rho}_T$ are defined in various ways following the described behaviour :

- Compliance behaviour in tension and shear:

$$\begin{aligned}\rho_N([u_N]) &= k_N([u_N] - [u_N]_0)\\ \boldsymbol{\rho}_T([u_T]) &= k_T[\boldsymbol{u}_T]\end{aligned}$$

  Coefficients, k_N and k_T, are the normal and the tangent stiffness of the interfaces. In small perturbations, $[u_N]$ and $[\boldsymbol{u}_T]$ are defined with respect to the initial local frame. $[u_N]$ must be slightly different from the gap function which is continuously updated with the local frame. The same method is applied to $[u_T]$. In large sliding, some cares must be taken in the use of this stiffness. In the latter, this stiffness will be associate with damage in order to reduce the effect of such behavior to small sliding. $[u_N]_0$ is set to the initial gap, usually considered as the equilibrium position if this gap measure a initial thickness of joint. A set of equivalent complementary equations may be written for Signorini-like constraints.:

$$\begin{aligned}g &\geq 0\\ r_N + k_N([u_N] - [u_N]_0) &\geq 0\\ g\,(r_N + k_N([u_N] - [u_N]_0)) &= 0\end{aligned}$$

  and for the Coulomb-like law:

  stick phase: if $\|\boldsymbol{v}_T\| = 0$ then
  $\|\boldsymbol{r}_T + k_T[\boldsymbol{u}_T]\| \leq \mu\,(r_N + k_N([u_N] - [u_N]_0))$

  slip phase: if $\|\boldsymbol{v}_T\| > 0$ then
  $\|\boldsymbol{r}_T + k_T[\boldsymbol{u}_T]\| = \mu\,(r_N + k_N([u_N] - [u_N]_0))$

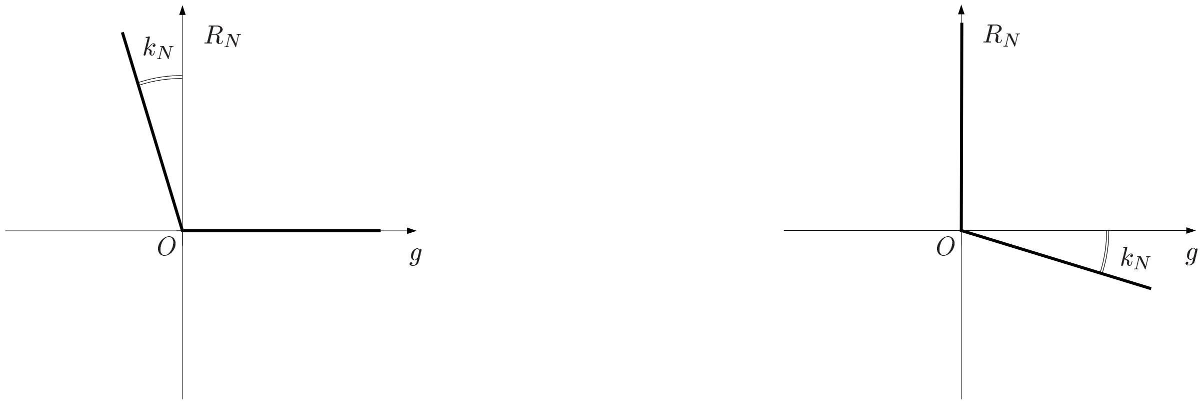
- Compliance behaviour in compression:
  As in tension, a flexibility behaviour may be defined in compression to represent stiffness of bodies or interfaces. This kind of flexibility is usually introduced in contact of rigid bodies. Auxiliary variables may be written as follows :

$$\tilde{g} = g + \frac{r_N}{k_N}$$

  which leads to a set of complementary equations :

$$g + \frac{r_N}{k_N} \geq 0 \qquad r_N \geq 0 \qquad (g + \frac{r_N}{k_N})r_N = 0$$

  This formulation is just considered in a formal way. Indeed, as we see in a further section, this flexibility is implemented in adding compliance term to reduced local stiffness matrix. Behaviour may be easily understood under the following form (see Figure):



(a) Compliance in compression

(b) Compliance in tension

Figure 2: Compliance models

contact	$g \leq 0$	$r_N = -k_N g$
no contact	$g > 0$	$r_N = 0$

- Brittle cohesion laws:
  In order to handle tensile strength in joints, a shifting may be introduced in the graph of Signorini:

  $$\rho_N([u_N]) = \alpha$$

  The parameter α is a tensile strength in tension. Its evolution may be driven by various laws. For masonry joints, a first binary representation has been carried out. a constant value is chosen for α. If a limit load was reached, tensile strength is set to zero and standard Signorini's condition is retrieved as interface laws. Coupling with compliance, presented above may be also considered. One obtain then a model of flexible interface with tension cutoff.

- Cohesive zone model with progressive interface damage model:
  Model of cohesive zone interface has been implemented in order to modelize progressive softening in joints. This model are inspired from cohesive laws with damage presented in[20][15]. Auxiliary variable may be written down as:

  $$\begin{aligned} \rho_N([u_N]) &= k_N \beta [u_N] \\ \rho_T([u_T]) &= k_T \beta [\boldsymbol{u}_T] \\ \dot{\beta} &= f(\beta, [u_N], [\boldsymbol{u}_T]) \end{aligned}$$

  Damage is taken into account by a scalar ratio β which holds in $[0, 1]$. This internal variable is driven by a complementary differential equation. For mortar joints, fractional function has been chosen as follows:

  $$b\dot{\beta} = -(w - (k_N[u_N]^2 + k_T[\boldsymbol{u}_T]^2)\beta)^-$$

  The stored energy w is the stored energy in the interface which represents a threshold for the damage evolution. b is a viscosity of a damage evolution and k_N and k_T are the stiffness of the interfaces.

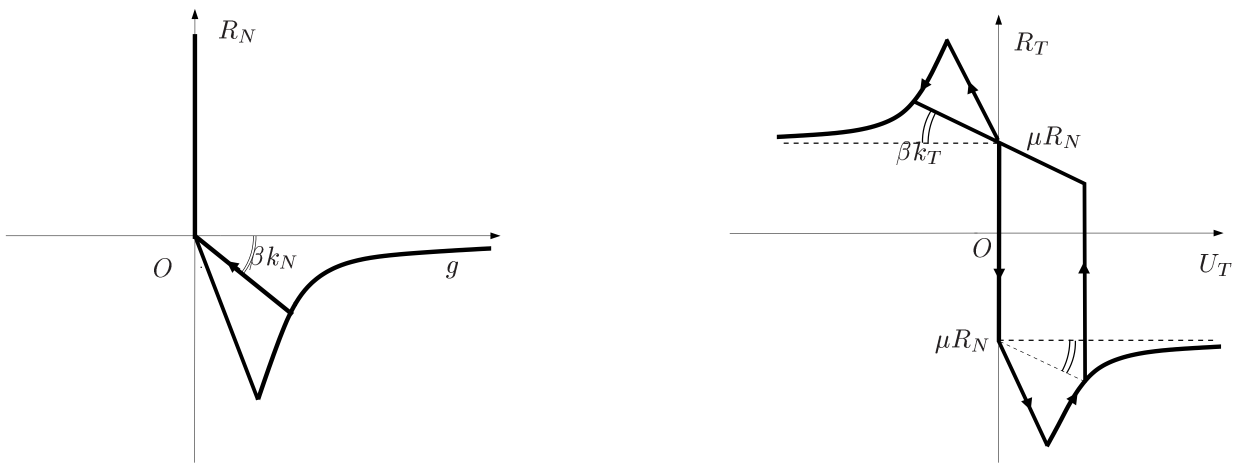
The choice are this various kind of interface model is driven by mechanical frame of the experiments. For instance, binary representation of a cohesion of an interface is often sufficient when we deal with large structure subjected to large dynamic loading such seismic excitations. In accurate quasi-static of more modest structure, enhanced law such cohesive zone model may be useful.

2.3 Time integration

The purpose of this part is to illuminate some of issues involved in time discretization and choices which have been made to deal with discontinuities. Equivalent forms of Equation (2) may be written under integral forms on a interval $[t_i, t_{i+1}]$ of length h :

$$\begin{aligned} \int_{t_i}^{t_{i+1}} \boldsymbol{M}(\boldsymbol{q}, t) d\boldsymbol{u} + \int_{t_i}^{t_{i+1}} \boldsymbol{Q}(\boldsymbol{u}, \boldsymbol{q}, t) & \\ -\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{q}, t)\, dt &= \int_{]t_i, t_{i+1}]} \boldsymbol{R}_\nu \, d\nu \\ \boldsymbol{q}(t_{i+1}) = \boldsymbol{q}(t_i) + \int_{t_i}^{t_{i+1}} \boldsymbol{u}(\tau)\, d\tau & \end{aligned}$$

Various computational strategies and time-stepping scheme are described in literature to integrate such system. A widespread technique introduce a time subdivision (t_k) over which the status of contact remains unchanged on any subinter-



(a) Normal law

(b) Tangent law

Figure 3: Cohesive zone model

val $[t_i, t_{i+1}]$. For instance, in the case of Signorini's condition, no contact gets loose or occurs on a subinterval of integration. Since this assumption is made, unilateral constraints may be explicitly expressed in term of bilateral constraints, that implies that inequalities inherent in unilateral constraints are replaced by some equalities. The kind of methods is qualified by Moreau as "Event-driven method" since time stepping is imposed by contact status. The major drawback of such method is that instant of changes in status of contact must be forecasted and computed. If the considered mechanical system contains a large number of contact pair, this forecasting becomes quite unfeasible. The approximation of the instant of contact are a expensive computational process. This calculation becomes quite inconceivable when we deal with a large number of contact which can occur simultaneously.

A major assumption of the NSCD method is to put the mean value impulse $\boldsymbol{R}(t_{i+1}) = \boldsymbol{R}_{i+1} = \dfrac{1}{h}\displaystyle\int_{]t_i,t_{i+1}]} \boldsymbol{R}_\nu d\nu$ as a primary unknown variable. The time subdivision is given a priori and no special contact time needs to be computed. The key-idea, hidden beyond this assumption is that only an average value $\boldsymbol{R}_{i+1}$ at the end of time step is considered to be relevant. Likewise, from a mathematical point of view, a measure description of the dynamic of a mechanical system subjected to impact may be contrasted with a distributional description. These two description are theoretically similar but the representation of contact by distributions, for instance Dirac distributions, implies the knowledge of contact time. Furthermore, this feature of the NSCD method allows to handle quasi-static evolution where many changes in contact status may occur in a subinterval.

A first linearization is made considering that the inertial operator have slow variations with respect of time. Then, it is assumed to be constant over the time interval and evaluated at $t^* = (1-\alpha)t_i + \alpha t_{i+1}, \boldsymbol{q}^* = (1-\alpha)\boldsymbol{q}(t_i) + \alpha\boldsymbol{q}(t_{i+1})$, $\alpha \in [0,1]$. Using the definition of $\boldsymbol{u}$, a differential system is obtained:

$$
\begin{aligned}
\int_{t_i}^{t_{i+1}} \boldsymbol{M}(\boldsymbol{q},t)d\boldsymbol{u} &= \boldsymbol{M}(\boldsymbol{q}^*,t^*)\int_{t_i}^{t_{i+1}} d\boldsymbol{u} \\
&= \boldsymbol{M}(\boldsymbol{q}^*,t^*)(\boldsymbol{u}(t_{i+1}) - \boldsymbol{u}(t_i)) \\
\boldsymbol{q}(t_{i+1}) - \boldsymbol{q}(t_i) &= \int_{t_i}^{t_{i+1}} \boldsymbol{u}(t)\,dt
\end{aligned}
$$

The proposed time-integration algorithm is a θ-method, mainly used in implicit form with $\theta \geq \dfrac{1}{2}$. Choice of a first order integration scheme instead of traditional integration algorithm like Newmark's method or HHT method is justified by the presence of strong non-linearities and discontinuities involved by unilateral contact. Indeed, higher order algorithm requires some conditions of regularity and continuity on the discretized variables like velocity, which can not be fulfilled in this kind of application. Contact analysis with higher order algorithm leads to severe instabilities [25]. Energy conserving algorithms, initiated by Simo and Laursen [14, 7] in contact friction problems, are builded on the same remarks on stability which produce similar algorithm.

Using usual notation ($\boldsymbol{x}(t_n) = \boldsymbol{x}_n$), a standard residual term is then obtain when θ-method may be applied:

$$
\begin{aligned}
\mathcal{R}(\boldsymbol{u}_{i+1}) =& -\boldsymbol{M}(\boldsymbol{q}^*,t^*)(\boldsymbol{u}(t_{i+1}) - \boldsymbol{u}(t_i)) \\
&+ \int_{t_i}^{t_{i+1}} -\boldsymbol{Q}(\boldsymbol{u},\boldsymbol{q},t) + \boldsymbol{F}(\boldsymbol{u},\boldsymbol{q},t)\,dt \\
&+ \int_{]t_i,t_{i+1}]} \boldsymbol{R}_\nu\,d\nu \\
=& -\boldsymbol{M}(\boldsymbol{q}^*,t^*)(\boldsymbol{u}(t_{i+1}) - \boldsymbol{u}(t_i)) \\
&-h\left\{(1-\theta)\boldsymbol{Q}_i + \theta\boldsymbol{Q}_{i+1}\right\} \\
&+h\left\{(1-\theta)\boldsymbol{F}_i + \theta\boldsymbol{F}_{i+1}\right\} + h\boldsymbol{R}_{i+1} \\
\text{with}\quad t^* =& (1-\alpha)t_i + \alpha t_{i+1} \\
\boldsymbol{q}^* =& (1-\alpha)\boldsymbol{q}(t_i) + \alpha\boldsymbol{q}(t_{i+1})
\end{aligned}
$$

$$\begin{aligned}\boldsymbol{q}(t_{i+1}) - \boldsymbol{q}(t_i) &= \int_{t_i}^{t_{i+1}} \boldsymbol{u}(t)\, dt \\ &= h\left\{(1-\theta)\boldsymbol{u}(t_i) + \theta \boldsymbol{u}(t_{i+1})\right\}\end{aligned}$$

Velocity is chosen as primary variable as conjugate variable of impulse of reactions forces, $\boldsymbol{R}_{i+1}$.

This non-linear equation is solved with a Newton method on $\mathcal{R}(\boldsymbol{u}_{i+1})$. $\boldsymbol{u}_{i+1}$ is obtained as the limit of the sequence, $\left(\boldsymbol{u}^k_{i+1}\right)_{k\in\mathbb{N}}$ given by the following linearization:

$$\mathcal{R}_L(\boldsymbol{u}^{k+1}_{i+1}) = \mathcal{R}(\boldsymbol{u}^k_{i+1}) + \left.\frac{\partial \mathcal{R}(\boldsymbol{u})}{\partial \boldsymbol{u}}\right|_{\boldsymbol{u}^k_{i+1}} (\boldsymbol{u}^{k+1}_{i+1} - \boldsymbol{u}^k_{i+1})$$

For simplicity's sake, slow variations with respect to $\boldsymbol{u}$ are assumed for the given applied load $\boldsymbol{F}$. Likewise, inertial operator $\boldsymbol{M}$, which has been assumed to be constant over the time interval is evaluated for $\alpha = 1$. This approximation is widely sufficient for our applications which are classified as "slow dynamics". This two assumption yields to:

$$\begin{aligned}\left.\frac{\partial \boldsymbol{F}(\boldsymbol{u},\boldsymbol{q},t)}{\partial \boldsymbol{u}}\right|_{\boldsymbol{u}^k_{i+1}} &\approx 0 \\ \left.\frac{\partial(\boldsymbol{M}(\boldsymbol{q}^*,t^*)(\boldsymbol{u}_{i+1}-\boldsymbol{u}_i))}{\partial \boldsymbol{u}}\right|_{\boldsymbol{u}^k_{i+1}} &\approx \boldsymbol{M}(\boldsymbol{q}^k_{i+1}, t_{i+1})\end{aligned}$$

Furthermore, one assumes that $\boldsymbol{R}_{i+1}$ is known and its value at iteration k , $\boldsymbol{R}^k_{i+1}$, is given by a external algorithm described in the section below. The Jacobian operator can be written down as:

$$\begin{aligned}\left.\frac{\partial \mathcal{R}_L(\boldsymbol{u})}{\partial \boldsymbol{u}}\right|_{\boldsymbol{u}^k_{i+1}} &= \boldsymbol{M}(\boldsymbol{q}^k_{i+1}, t_{i+1}) \\ &+ h\theta\left\{\boldsymbol{C}_t(\boldsymbol{u}^k_{i+1}, \boldsymbol{q}^k_{i+1}, t_{i+1})\right\} \\ &+ h\theta\left\{h\theta \boldsymbol{K}_t(\boldsymbol{u}^k_{i+1}, \boldsymbol{q}^k_{i+1}, t_{i+1})\right\}\end{aligned}$$

$\boldsymbol{K}^k_t(\boldsymbol{u}^k_{i+1}, \boldsymbol{q}^k_{i+1}, t_{i+1}) = \left.\frac{\partial \boldsymbol{Q}(\boldsymbol{u},\boldsymbol{q},t)}{\partial \boldsymbol{q}}\right|_{\boldsymbol{u}^k_{i+1}}$ and $\boldsymbol{C}^k_t(\boldsymbol{u}^k_{i+1}, \boldsymbol{q}^k_{i+1}, t_{i+1}) = \left.\frac{\partial \boldsymbol{Q}(\boldsymbol{u},\boldsymbol{q},t)}{\partial \boldsymbol{u}}\right|_{\boldsymbol{u}^k_{i+1}}$ are respectively the tangent stiffness and damping operator.

Linearized equations lead to following algorithm :

$$(\boldsymbol{M}^k + h\theta \boldsymbol{C}^k_t + h^2\theta^2 \boldsymbol{K}^k_t)(\boldsymbol{u}^{k+1}_{i+1} - \boldsymbol{u}^k_{i+1}) = \mathcal{R}(\boldsymbol{u}^k_{i+1})$$

$$\begin{aligned}\mathcal{R}(\boldsymbol{u}^k_{i+1}) &= -\boldsymbol{M}^k(\boldsymbol{u}^k_{i+1} - \boldsymbol{u}_i) - h\left\{(1-\theta)\boldsymbol{Q}_i + \theta \boldsymbol{Q}^k_{i+1}\right\} \\ &+ h\left\{(1-\theta)\boldsymbol{F}_i + \theta \boldsymbol{F}^k_{i+1}\right\} + h\boldsymbol{R}^k_{i+1} \\ \boldsymbol{q}^{k+1}_{i+1} &= \boldsymbol{q}_i + h(1-\theta)\boldsymbol{u}_i + h\theta \boldsymbol{u}^{k+1}_{i+1}\end{aligned}$$

Let us introduce a particular velocity to simplify the form of the above algorithm. The so-called free velocity is the solution at the step $k+1$ of the algorithm when no reaction force is taken into account. This velocity is denoted by $\boldsymbol{u}^{k+1}_{free}$ and is given by

$$\begin{aligned}\boldsymbol{u}^{k+1}_{free} &= \boldsymbol{u}^k_{i+1} + (\boldsymbol{W}^k)\mathcal{R}_{free}(\boldsymbol{u}^k_{i+1}) \\ \mathcal{R}_{free}(\boldsymbol{u}^k_{i+1}) &= -\boldsymbol{M}^k(\boldsymbol{u}^k_{i+1} - \boldsymbol{u}_i) - h\left\{(1-\theta)\boldsymbol{Q}_i + \theta \boldsymbol{Q}^k_{i+1}\right\} \\ &+ h\left\{(1-\theta)\boldsymbol{F}_i + \theta \boldsymbol{F}^k_{i+1}\right\} + h\boldsymbol{R}^k_{i+1}\end{aligned}$$

with the following notation for inverse of the iteration matrix: $\boldsymbol{W}^k = (\boldsymbol{M}^k + h\theta \boldsymbol{C}^k_t + h^2\theta^2 \boldsymbol{K}^k_t)^{-1}$

Basic integration algorithm of the dynamical equation may be summarized as

$$\begin{aligned}\boldsymbol{u}^{k+1}_{i+1} &= \boldsymbol{u}^{k+1}_{free} + \boldsymbol{W}^k\left\{h\boldsymbol{R}^k_{i+1}\right\} \\ \boldsymbol{q}^{k+1}_{i+1} &= \boldsymbol{q}_i + h(1-\theta)\boldsymbol{u}_i + h\theta \boldsymbol{u}^{k+1}_{i+1}\end{aligned}$$

3 Solving unilateral constraints

In the previous section, numerical algorithms are derived from dynamical equations with no detail on the computation of reaction forces. In this part, numerical ways to solve unilateral constraints in a consistent way with the time-stepping are presented. Following items are key-point in the derivation of an algorithm:

- Solving in local frame that requires a condensation to local variable of system.
- Time discretization of graph
- Solving trough Gauss-Seidel like algorithm that iterates on each contact pair.

3.1 Reduction and condensation to local variables

In order to solve the unilateral constraints in local frame, linear mappings $\boldsymbol{H}^\alpha$, ${}^t\boldsymbol{H}^\alpha$ are introduced for each contact pair α which associate local variable $\boldsymbol{v}^\alpha, \boldsymbol{r}^\alpha$ defined at locus of contact to generalized variables $\boldsymbol{u}^\alpha, \boldsymbol{R}^\alpha$ such that:

$$\boldsymbol{v}^\alpha = {}^t\boldsymbol{H}^\alpha(\boldsymbol{q})\boldsymbol{u}^\alpha \qquad \boldsymbol{R}^\alpha = \boldsymbol{H}^\alpha(\boldsymbol{q})\boldsymbol{r}^\alpha$$

Two types of transformations are taken into account through $\boldsymbol{H}^\alpha\, {}^t\boldsymbol{H}^\alpha$:

- Orthogonal transformation which corresponds to standard basis changes and kinematic relations between local and global frames.
- Transformations between the actual locus of contact and the generalized variables such as, kinematic constraints for rigid bodies, due to transport of torques and the transposed transport of velocity. For discretized deformable bodies, spatial interpolation is generally given by the interpolation of finite element between values of forces and velocities at nodes and values at locus of contact. Interpolation on surfaces where contact may occur is given by reduction of finite element interpolation on the surfaces.

$\boldsymbol{H}^\alpha$ may be seen on one hand as a condensation to variables concerned by the contact interaction and the other hand as kinematic constraints which are introduced through basis changes. Further to the assembling contribution of each contact interaction formally denoted as a sum operation, discretized dynamical equation may be rewritten as follows in terms of local variables (subscript $i+1$ in the sequel is omitted for simplicity's sake):

$$\boldsymbol{v}^{\alpha^{k+1}} = \boldsymbol{v}^{\alpha^{k+1}}_{free} + \sum_{\beta} h \boldsymbol{w}^{\alpha\beta} \boldsymbol{r}^{\beta^{k+1}}$$

with the reduced inverse of the iteration matrix $\boldsymbol{w}^{\alpha\beta} = {}^t\boldsymbol{H}^{\alpha}(\boldsymbol{q})\boldsymbol{W}^k\boldsymbol{H}^{\beta}$. $\boldsymbol{w}^{\alpha\beta}$ is called the influence matrix and represents the global response of the structurefor a given reaction force.

This kind of reduction is usually made when solving of unilateral constraints is made in local frame and it's usually a expensive operation in which a cumbersome inverse must be explicitly computed. In our application, we take advantage of the block structure of iteration matrix. Indeed, the assumed discrete nature of the structure involves a mere splitting of the matrix into blocks corresponding to each floating body. Inversion of each block iteration matrix is cheap. Therefore, implementation of various algorithm have been made in taking into account the discrete nature of the structure.

3.2 Time discretization of graphs

In spite of many efforts made in the time discretization of velocity in order to integrate the dynamical equation, approximation of unilateral constraints still remains a tenuous problem. If Signorini's condition is first considered, a ingenuous way to discretize is to ensure that condition must fulfilled for g^{k+1}, so that :

$$g^{k+1} \geq 0 \qquad r_N^{k+1} \geq 0 \qquad g^{k+1} r_N^{k+1} = 0$$

We have seen that relative velocity may be viewed as the time derivative of the gap function: $\dot{g} = \boldsymbol{v}_N$. Time approximation of this equation leads to:

$$g^{k+1} = g_i + h(1-\theta)(\boldsymbol{v}_N)_i + h\theta \boldsymbol{v}_N^{k+1}$$

which can be written in a equivalent form in order to express Signorini's condition in terms of relative velocity which is our primary unknown:

$$\begin{aligned} \frac{1}{h\theta} g^{k+1} &= \boldsymbol{v}_N^{k+1} - \tilde{g}_i \\ \text{with} \quad \tilde{g}_i &= -\frac{1}{h\theta} g_i + (1 - \frac{1}{\theta})(\boldsymbol{v}_N)_i \end{aligned}$$

The above discretization yields to a velocity formulation of a discretized condition as follows:

$$\boldsymbol{v}_N^{k+1} - \tilde{g}_i \geq 0 \quad \boldsymbol{R}_N^{k+1} \geq 0 \quad \boldsymbol{R}_N^{k+1}(\boldsymbol{v}_N^{k+1} - \tilde{g}_i) = 0$$

Some remarks can be made on this formulation of the time discretized Signorini's condition. Indeed, relative velocity and gap are linked through this equation which involves some troubles when contact have to be conserved during time. A consistent discretization should provide some properties particularly when contact has to be persistent within the time interval. For instance, consistency involves that following constraints must be fulfilled:

$$\left(g_i = 0 \quad \text{and} \quad g^{k+1} = 0\right) \Longrightarrow \boldsymbol{v}_N^{k+1} = 0$$

This property may be easily ensured if discontinuous and independent fields are used. But in our case, if $g_i = 0$ and $g^{k+1} = 0$, $\boldsymbol{v}_N^{k+1}$ have to be equal at the same time at $(1-\frac{1}{\theta})(\boldsymbol{v}_N)_i$. Major drawbacks, generated by such approximations, are oscillatory artifacts of contact velocity.

In order to reach compatibility between gap and velocity approximation, other approximations are chosen for the gap value. First of these methods is an Euler-backward approximation such that $g^{k+1} = g_i + h\boldsymbol{v}_N^{k+1}$. With such approximation, $\boldsymbol{v}_N^{k+1}$ can vanish if gap remains equal to zero within the time interval. A another way to overcome this difficulty is to choose a shifted time discretization for gap. Proposed time approximation for gap has been given by Jean [13] :

$$\begin{aligned} \bar{g}_i &= g_i + h(1-\theta)(\boldsymbol{v}_N)_i \\ \bar{g}^{k+1} &= g^{k+1} + h(1-\theta)\boldsymbol{v}_N^{k+1} \end{aligned}$$

and an Euler-backward like approximation is inferred for $\bar{g}^{k+1}$:

$$\bar{g}^{k+1} = \bar{g}_i + h\boldsymbol{v}_N^{k+1}$$

This approximation ensures consistency property for the couple $(\bar{g}, \boldsymbol{v}_N)$ and Signorini's condition may be rewritten as:

$$\bar{g}^{k+1} \geq 0 \qquad r_N^{k+1} \geq 0 \qquad \bar{g}^{k+1} r_N^{k+1} = 0$$

Furthermore, if the contact is forecasted within the time interval, velocity formulation of Signorini's condition may be used:

$$\boldsymbol{v}_N^{k+1} \geq 0 \qquad r_N^{k+1} \geq 0 \qquad \boldsymbol{v}_N^{k+1} r_N^{k+1} = 0$$

Time discretization of Coulomb's law is a less vexed question. Indeed, Coulomb's friction is written in velocity term in a natural way. Therefore, a proposed discretized form of this constraints is given by:

$$(\boldsymbol{v}_T^{k+1}, \boldsymbol{r}_T^{k+1}) \in GR - Coulomb(r_N^{k+1})$$

3.3 Gauss-Seidel-like algorithm

Solving of unilateral constraints is made by means of iterative method over the set of contact pair. For each contact pair α, constraints are solved giving for the other contact pair $\beta \neq \alpha$ provisional values. This provisional values are updated within the loop over contact pair.

For each contact pair α, following system have to be solved (superscript k+1 is omitted in the sequel):

$$\begin{aligned} \boldsymbol{v}^{\alpha} &= \boldsymbol{v}^{\alpha}_{free\alpha} + h\boldsymbol{w}^{\alpha\alpha}\boldsymbol{r}^{\alpha} \\ \boldsymbol{v}^{\alpha}_{free\alpha} &= \boldsymbol{v}^{\alpha}_{free} + \sum_{\beta\neq\alpha} h\boldsymbol{w}^{\alpha\beta}\boldsymbol{r}^{\beta} \\ (\bar{g}^{\alpha}, r_N^{\alpha}) &\in GR - Signorini \\ (\boldsymbol{v}_T^{\alpha}, \boldsymbol{r}_T^{\alpha}) &\in GR - Coulomb(r_N^{\alpha}) \end{aligned}$$

In 2 dimensional case or in particular 3 cases like frictionless constraints, solution for a pair α is carried out by simple intersection of graphs. This intersection provides an analytical solution to the unilateral constraints with friction. From a computational point of view, this method constitutes a very efficiency ways to solve the set of constraints. Details on this solving is presented in detail in [13].

Likewise, the problem of frictionless constraints may be treat with standard optimization algorithm in general tridimensionnal case. When Coulomb's Friction law is introduced, many problems raises from the dependence of the sliding threshold to the pressure. In the NSCD method, solution is performed by means of generalized newton method. In few words, a quasi-augmented Lagrangian is introduced trough a mixed penalty-duality method that leads to following formulation of the constraints:

$$\begin{aligned} \boldsymbol{v}^\alpha &= \boldsymbol{v}^\alpha_{free\alpha} + h\boldsymbol{w}^{\alpha\alpha}\boldsymbol{r}^\alpha \\ r^\alpha_N &- proj_{\mathbb{R}^+}(r^\alpha_N - \lambda_N \bar{g}^\alpha) = 0 \\ \boldsymbol{r}^\alpha_T &- proj_{\hat{C}(\mu(r_N - \rho\bar{g}^\alpha))}(r^\alpha_T - \lambda_T \boldsymbol{v}^\alpha_T) = 0 \end{aligned}$$

$\lambda_{N,}\lambda_T$ are nonnegative 'penalty' factor and $\hat{C}(\mu(r_N - \rho\bar{g}^\alpha))$ is build prolongating the cone $\hat{C}(\mu r_N)$ by the negative half-line. The latter system of equation may be summarized in a formal way:

$$\Phi(\bar{g}^\alpha, \boldsymbol{v}_T, r_N, \boldsymbol{r}_T) = 0$$

Φ is a continuous operator, raywise linear and using the definition of generalized Jacobian for Lipschitz continuous function, roots of this system may be solved by the Newton method [3].

4 Quasi-static equilibrium

Obtaining a quasi-static equilibrium, when floating structure are encountered, is not a trivial task. Since masonry structure is considered as a collection of bodies interconnected with unilateral constraints, no explicit boundary condition is applied to bodies. The existence of equilibrium state is governed by the computation of a set of reaction forces, which vanishes the total external forces torque. At initial state when reactions are unknown, rigid body motions are not defined. Therefore the problem of a quasi-static evolution of floating structure is a ill-posed problem. The fact that reactions forces are computed quantities that depend implicitly from the generalized variables through multi-valued operator makes hard the computation of an a priori equilibrium state.

An usual way to deal with such system is to consider a equivalent dynamical problem. Despite no boundary conditions are prescribed, a dynamical problem remains a mathematically well-posed problem. From a numerical point of view, dynamical iteration matrix $(\boldsymbol{M}^k + h\theta \boldsymbol{C}^k_t + h^2\theta^2 \boldsymbol{K}^k_t)$ is still a definite positive matrix and its inversion is always possible, whatever the time step. On the other hand, internal forces are by definition equal to zero for any rigid body motion. Therefore, stiffness tangent matrix $\boldsymbol{K}^k_t$ have zero eigenvalues which corresponds to linearized rigid body motion. This singular value prevents any quasi-static incremental computation since iteration matrix is not invertible. The mass matrix $\boldsymbol{M}^k$, as a definite positive matrix, regularizes the incremental algorithm and allow to take into account rigid body motion. As in buckling modelling or perfect plastic constitutive behavior, this feature can be qualified of dynamic regularization

The major problem raised by the approach is the sensitivity of results to inertial effects. If the uniqueness of solution is not ensured, dynamical evolution represent a way to choose a particular path to reach an equilibrium, among various ways to reach an other one. For instance, let us consider the equilibrium of a dry bond stone wall submitted to gravity load. At initial time, the considered structure is free stressed and rigid body motion are undefined for the most of blocks. If dynamical evolution is considered, elastic waves causes oscillations that vanish under the effect of numerical and/or natural damping. Cracks in head joints must be appear as the result of this oscillations owing to the introduction of artificial dynamical effects introduced.

To overcome this difficulty, a standard method is to use of damping, that minimize dynamical effects and the elastic wave propagation. Damping may be introduced by means of material or numerical viscosity. These viscosities are often hardly to size. Moreover, effects of damping on the way to chose a loading path is not well evaluated. An other method may be cited consisting of regularizing mutli-valued graph with non-linear compliances. These compliances are introduced directly in iteration matrix and define rigid body motion in providing boundary conditions. Like viscosity introduction, this method suffers from lack of definition of local compliance of interfaces and make the problem very ill-conditioned.

Three ways are proposed to tackle this problem :

- First, the use of a dynamical approach with an artificial damping introduce by setting velocity to zero further to each time iteration. Dynamical algorithm is used in a standard way with small time step with the respect to fundamental period of the structure but at each time step, velocity is put equal to zero to minimize dynamical effects. Even if this way to reach an equilibrium is debatable, it allow to obtain one state of equilibrium where dynamical effects has been omitted.

- A other way is to take advantage to the structure of the dynamical iteration matrix which is always invertible for all time step. If the chosen time increment is larger than a fundamental modal period, numerical damping run as filter with the respect to these frequencies. The integration algorithm degenerates in a quasi-static incremental solver. The decrease of the predominance of the mass matrix in the iteration matrix when the time step increase is a numerical justification of this property of the algorithm. The θ-method algorithm is unconditionally stable for $1 \geq \theta \geq \dfrac{1}{2}$ in linear cases. In the presence of unilateral constraints, it still remains very stable particularly with θ equal to 1, performing the maximum numerical damping. A drawback of this approach is the loss of conditioning of the iteration matrix that have many effects on the rate of convergence of the solver of unilateral constraints. As the tangent stiffness matrix became preponderant in the iteration matrix, its ill-conditioning is imposed to the solver which have many difficulties to find a increment of rigid displacement to reach a quasi-static equilibrium. This problem increase with the number of contact pair and so the number of reactions forces, which enter in the external forces to each bodies. Indeed, the lack of information on the rigid body motion and how the increment of rigid body motion must be made from the initial state to the equilibrium one, involves severe perturbations on the algorithm if number of contact pair is high. The solver is based on an iteration on different contact pair. When any pair have to be compute, a large rigid body motion is found as the result of the reaction given by the algorithm which try to answer to the initial large free velocity. This rough predictive solution due to indeterminate rigid body motion involves a slow rate of convergence of the algorithm

- The last which is proposed to reach equilibrium state is based on a Singular Value Decomposition in order to extract rigid body motion of the iteration matrix. This extraction is made by means of a projection of the set generalized variable on two dual space which one corresponds to the null space of tangent stiffness matrix. Projection are defined with a scalar product defines upon mass matrix. This approach improves the rate of convergence of the constraints solver but do not solve the entire problem of the rigid body motion.

Many question may be asked for the formulation of the problem of obtaining an equilibrium state with floating structures: What is the relevance of a model where the structure is considered as a set of floating structure in a initial state? How to give more informations to the mechanical system in order to define the rigid body displacement useful to reach a equilibrium state? The masonry structure is built in many stages and effects of gravity causes stress in the structure appears at each stages of its construction. This stages cannot be modelized in numerical simulation its whole integrity. Furthermore, major stages of history of construction is unknown but may be important in the process which have lead to the actual structure.

5 Numerical applications

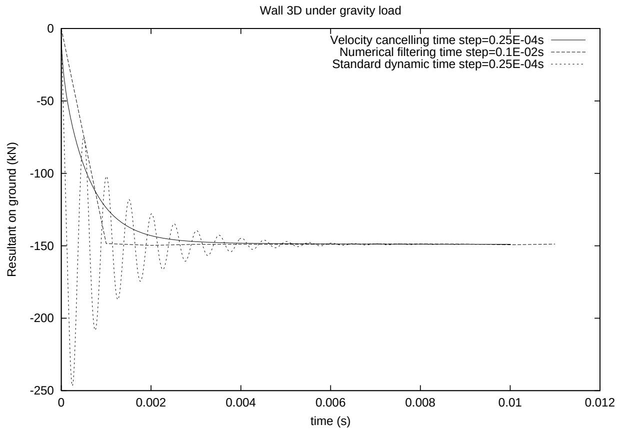
In this section, some examples of NSCD method applied to masonry structure are provided. These applications are chosen for their didactic aspects more than for their realistic one. A small wall under gravity load is considered as first of these examples. An experiment of settlement of ground is made to show the ability of the method to treat as well as small strains in a accurate way, than large strains and stresses occurring when large fracture and sliding are considered. In a second example, a larger structure is introduced, enabling to handle genuine tridimensional effects.

5.1 Masonry wall

The specimen consists in a wall with an width/length ratio of $\frac{1}{2}$, made up of 12 running courses of regular blocks (dimensions: $250 \times 500 \times 300 mm^3$) and $25mm$ thick mortar. Material properties are chosen for stone blocks as: $E = 60\, GPa, \nu = 0.27, \rho = 2700 kg/m^3$. Mortar is just considered as a frictional interface with $\mu = 0.5$. Likewise, Boundary conditions on ground are given by means of frictional interfaces with $\mu = 1$. The wall is subjected to a distributed body load $g = 9.81 m/s^2$ from an initial free stressed state. Each block is meshed by 8 H8 (hexahedral linear finite element with 8 nodes). On each face, 4 contact pair are considered to integrate interface models.

The aim of the first test is to obtain a quasi-static equilibrium. Several method, described in the section before have been used to reach this state with floating bodies. As depicted in Figure 6, where resultant on ground is plotted versus time, dynamical effects and elastic waves propagation are taken into account in quite different ways. Damping, introduced by setting velocity to zero or numerical filtering implies equilibrium obtained as an approximation of a quasi-static evolution. For standard dynamic and computation in which velocity vanishes at each iteration, time step is set to a fraction of the fundamental modal period T_{mod}, which is approximately equal to $0.5.10^{-3}s$. In this test, time step is set to $T \mod /20 \approx 0.25.10^{-4}s$. When numerical filtering is considered, time step is set to $2T_{mod} \approx 0.1.10^{-2}s$.

Figure 6 : Resultant on ground versus time



On Figure 4(b) and Figure 4(a), two various equilibrium are illustrated by contour of Mises stresses on deformed shape. First, dynamical regularization causes artificial cracks in head joints due to oscillations. In second case, deformed shape is considered to be more relevant of a reliable quasi-static loading path.

From this latter equilibrium state, a settlement of ground has been performed. This ground motion consists in 50 mm height of an half of the wall. This loading is carried out by discarding velocity at each time step to reduce waves propagation. Deformed shape is given on Figure 7.

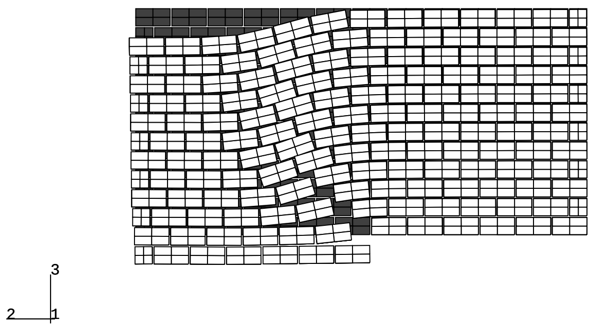
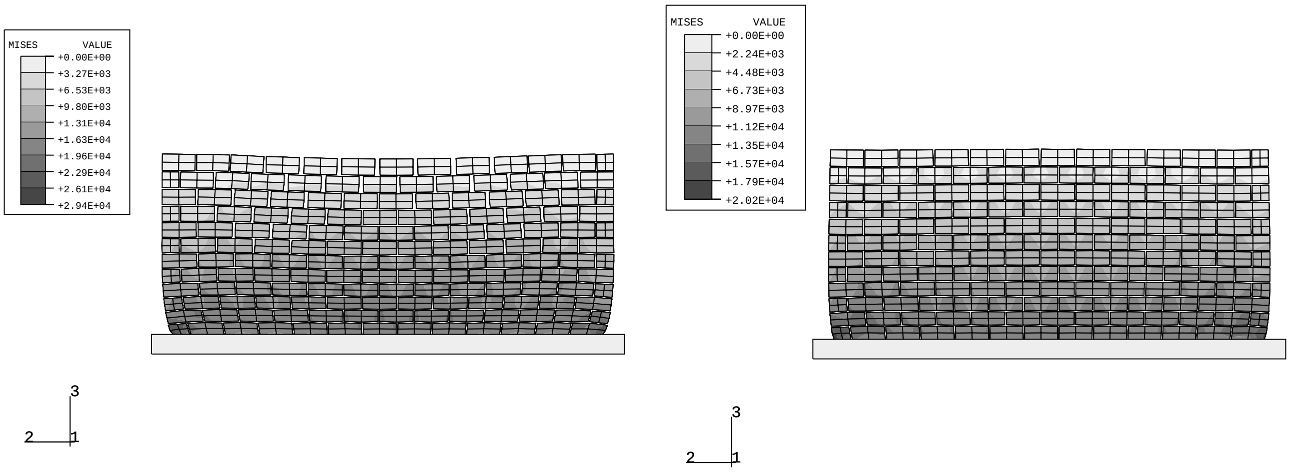


Figure 7 : Settlement of ground



(a) Mises stresses with standard dynamic computation

(b) Mises stresses with cancelling velocity

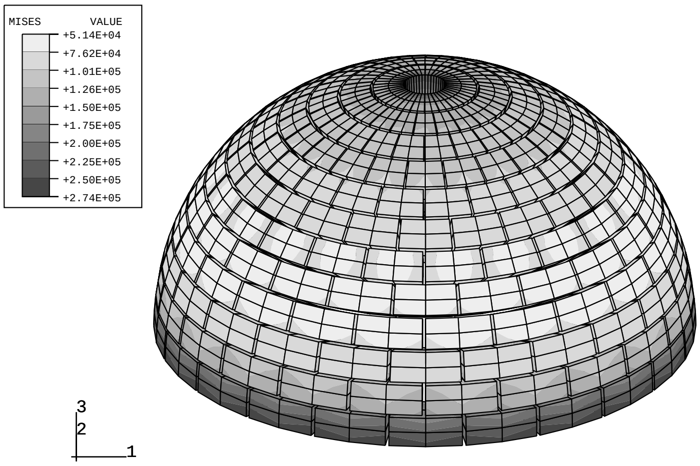
Figure 4: Wall under gravity load (magnification=2.5E+06)

5.2 Dome

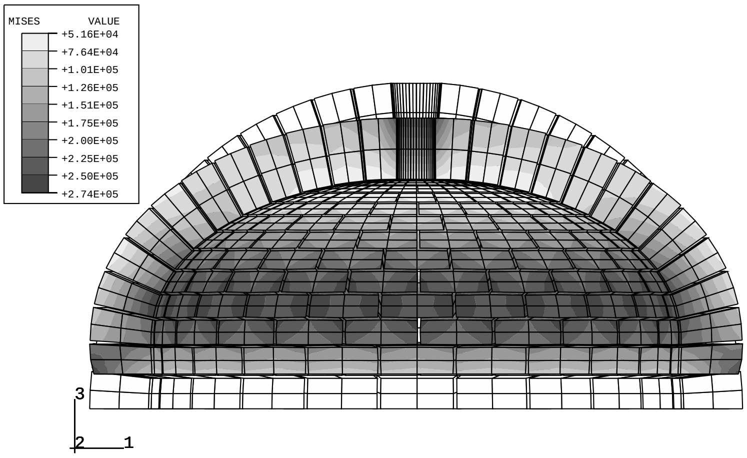
In this test, the considered specimen of dome is composed of (12 $\times$ 24 blocks meshed with 8 H8). Material properties are set at the same values as in the section before. Interface model is chosen as frictional model with brittle cohesion. Cohesion threshold is chosen as $1000N$. Figure 5(a) shows Mises stresses under gravity load. On Figure 5(b), it's noteworthy that stresses distributions in a clip-plane, taken as a symmetry plane, follows prediction of thrust line described in Heyman[11]. More complicated loading experiments might be considered but this kind of simulations would require more realistic conditions.

6 Conclusion

A reliable numerical method has been presented in this paper to deal with divided structures. Taking advantage of locus of failure in masonry structures, this method is based on micro-modelling by means of interface model. Non-smooth treatment is used to solve set of equations with unilateral constraints, that model basic response of interfaces. This method has been applied on masonry structure, e.g., wall and dome. A line of research, in collaboration with GAMSAU/MAP UMR CNRS 694 [1] consists in an implementation of a tool, which should be designed to encompass problem from the stereophotogrammetric restitution of geometrical shape to the numerical modelling. This work will enable to consider realistic geometry and mechanical problems in their contexts.



(a) Mises stresses under gravity load



(b) Mises stresses in clip-plane (magnification=2.5E+06)

Figure 5: Dome under gravity load

References

[1] V. Acary, J.Y. Blaise, P. Drap, M. Florenzano, S. Garrec, M. Jean, and D. Merad. NSCD method applied to mechanical simulation of masonry in historical buildings using MOMA. In *XVII CIPA International Symposium*. CIPA (International Committee for Architectural Photogrammetry), Olinda, Brazil, October, 3-6 1999.

[2] V. Acary and M. Jean. Numerical simulation of monuments by the contacts dynamics method. In DGEMN-LNEC-JRC, editor, *Monument-98, Workshop on seismic perfomance of monuments*, pages 69–78, November 12-14 1998.

[3] P. Alart and A. Curnier. A mixed formulation for frictional contact problems prone to newton like solution method. *Comput. Methods Appl. Mech. Engrg.*, 92(3):353–375, 1991.

[4] A. Anthoine. In-plane behavior of masonry: a literature review. Technical Report EUR 13840 E, Commission of the European communities, 1992.

[5] G. Armand, M. Boulon, J.C. Jay, C. Papadopoulos, G. Picaud, M. Reveillet, B. Rey, and J.F. Verdys. Monuments under seismic action. Task number 3 - Interface Properties EV5V-CT93-0300, Laboratoire Sols Solides Structures IMG INPG CNRS, 1997.

[6] J.-L. Chaboche, F. Feyel, and Y. Monerie. Interface debonding models : a viscous regularization with a limit rate dependency. *Int. J. Solids Structures*, 1999. to be published.

[7] V. Chawla and T.A Laursen. Energy consistent algorithms for frictional contact problem. *Int. J. for Numer. Meth. Engng.*, 42, 1998.

[8] P.A. Cundall and O.D.L. Strack. A discrete numerical model for granular assemblies. *Geotechnique*, 29(1):47–69, 1979.

[9] R. De Borst, L.J. Sluys, H.B. Mülhaus, and J. Pamin. Fundamental issues in finite element analyses of localization of deformation. *Engineering Computations*, 10:99–121, 1993.

[10] M. Fremond and B. Nedjar. Damage, gradient of damage and principle of virtual power. *Int. J. Solids Structures*, 33(8):1083–1103, 1996.

[11] J. Heyman. *The Stone Skeleton, Structural Engineering of masonry architecture*. Cambridge University Press, 1995.

[12] M. Jean. Brittle fracture of a cohesive composite. In *Instabilité de frottement*. Laboratoire de Mathématiques - Université de Savoie, September 27-29 1999.

[13] M. Jean. The non smooth contact dynamics method. *Comput. Methods Appl. Mech. Engrg.*, 177:235–257, 1999. Special issue on computational modeling of contact and friction, J.A.C. Martins and A. Klarbring, editors.

[14] T.A. Laursen and J.C. Simo. A continuum-based finite element formulation for the implicit solution of multybody, large deformation frictional contact problems. *Int. J. for Numer. Meth. Engng.*, 36:3451–3485, 1993.

[15] Y. Monerie and M. Raous. A model coupling adhesion to friction for the interaction between a crack and a fiber/matrix interface. Zeitschrift für Angewandte Mathematik und Mechanik, Special issues “Annual Gesellschaft für Angewandte Mathematik und Mechanik Conference”, April 12-16 1999.

[16] J.J. Moreau. Unilateral contact and dry friction in finite freedom dynamics. In J.J. Moreau and P.D. Panagiotopoulos, editors, *Non smooth mechanics and applications*. Springer Verlag, 1988.

[17] J.J. Moreau. Some basics of unilateral dynamics. In F. Pfeiffer, editor, *IUTAM Symposium on Multibody dynamics*. Kluwer, August 3-7 1998.

[18] M. Ortiz and A. Pandolfi. Finite-deformation irreversible cohesive elements for three-dimensional crack propagation analysis. *Int. J. for Numer. Meth. Engng.*, 44:1267–1282, 1999.

[19] P. Pegon and A. Anthoine. Numerical strategies for solving continuum damage problems with softening: application to the homogenization of masonry. *Computers and Structures*, 64(1-4):623–642, 1997.

[20] M. Raous, L. Cangémi, and M. Cocu. A consistent model coupling adhesion, friction, and unilteral contact. *Comput. Methods Appl. Mech. Engrg.*, 177:383–399, 1999. Special issue on computational modeling of contact and friction, J.A.C. Martins and A. Klarbring, editors.

[21] M. Save. On the application of limit analysis to masonry structures. In Lemaire R.M. and Van Balen K., editors, *Stable or unstable ? - Structural consolidation of ancient buildings*, pages 231–248. Center for conservation of historic Towns and buildings, Leuven University Press, April 12-16 1986.

[22] A. A. Shabana. *Dynamics of Multi-body Systems*. Cambridge University Press, second edition, 1998.

[23] G.H. Shi. *Deformation discontinuous analysis, a new numerical model for the statics and dynamics blocks analysis*. PhD thesis, Department of Civil engineering, university of California at Berkeley, 1988.

[24] P.M. Suquet. *Plasticité et homogénéisation*. Thèse d'état de l'Université Pierre et Marie Curie, Paris VI, 1982.

[25] D. Vola, E. Pratt, M. Jean, and M. Raous. Consistent time discretization for dynamical frictional contact problems and complementarity techniques. *Revue Européenne des Eléments Finis*, 7(1-2-3):149–162, 1998.