An overview of Non Smooth Dynamical Systems Higher order systems, numerical methods and links with Optimization

Vincent Acary, BipOp Project, INRIA Rhône–Alpes

vincent.acary@inrialpes.fr

Oberwolfach, January 13, 2005

Activities of the BipOp Project. http://www.inrialpes.fr/bipop

- Team: 4 Permanent members + 5 PhD students + 1 post doc
  - Bernard Brogliato (Head of the Project)
  - Claude Lemaréchal
  - Pierre–Brice Wieber
  - Vincent Acary
- The core of our activities is in the field of the Non-Smooth Analysis :
  - Non smooth optimization (CL)
  - Modeling of Non Smooth Dynamical Systems (NSDS) (VA, BB)
  - Control of NSDS (PBW, BB)
  - Simulation of NSDS (VA, PB)
- Favorite applications :
  - Mechanical systems with contact and friction (Multi-body dynamics, Granular materials, Buildings made of masonry, ..) with possibly real-time constraints (Haptic feedback)
  - Electrical networks with idealized components (Diodes, transistors, switch, ...)
  - Walking robot and bipedal Locomotion
- European Projects :
  - FP5 project SICONOS coordinated by B. Brogliato.
    ➜ Main outcome : Open source software platform for simulation, modeling and control of NSDS (Python, C++, F77) `http://siconos.inrialpes.fr/software`
  - The FP6 Network of Excellence HyCon (Hybrid Control)

Outline

What is a Non Smooth Dynamical System (NSDS) ?

A NSDS is a dynamical system is characterized by two correlated features :

- a non smooth evolution with the respect to time, for instance :
  - Jumps in the state and/or in its derivatives wrt. time
  - Generalized solutions (distributions)
- a set of non smooth laws (Generalized equations) between the state x and a set of Lagrange multipliers λ

A typical example is the finite-dimensional unilateral dynamics :

$$\begin{cases} \dot{x} = f(x,t) + \lambda, x \in \mathbb{R}^n \\ x \geq 0, \lambda \geq 0, x.\lambda = 0 \end{cases} \quad (1)$$

that can be written as a (unbounded) differential inclusion :

$$-\dot{x} + f(x,t) = -\lambda \in \partial\Psi_{\mathbb{R}_+}(x) = \mathcal{N}_{\mathbb{R}_+}(x) \quad (2)$$

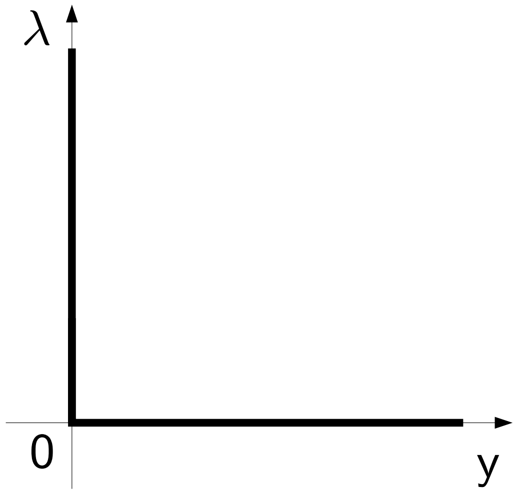
where - $\Psi_{\mathbb{R}_+}$ is the indicatrix function of $\mathbb{R}_+$

- $\mathcal{N}_{\mathbb{R}_+}$ the normal cone to $\mathbb{R}_+$, i.e, $\mathbb{R}_-$

Typical examples

- Linear Complementarity Systems (LCS)

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \leq y \perp \lambda \geq 0 \end{cases} \tag{3}$$



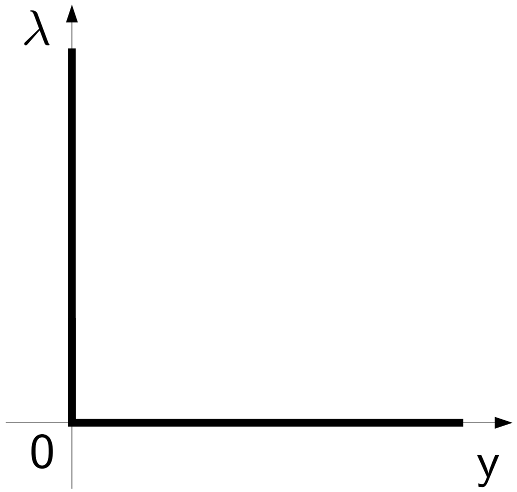
with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$, for m constraints.

Typical examples

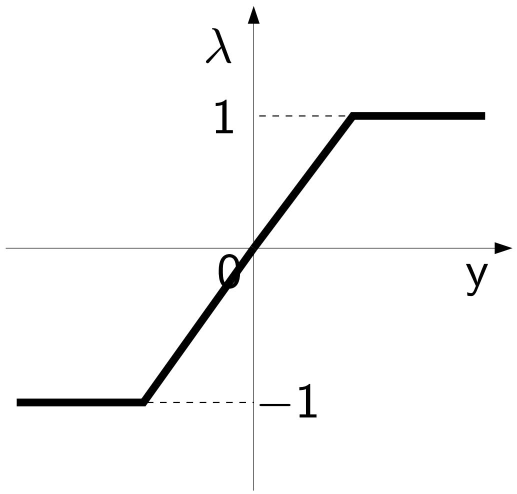
Linear Complementarity Systems (LCS)

$$\begin{cases} \dot{x} = Ax + B\lambda, & x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \leq y \perp \lambda \geq 0 \end{cases} \quad (3)$$

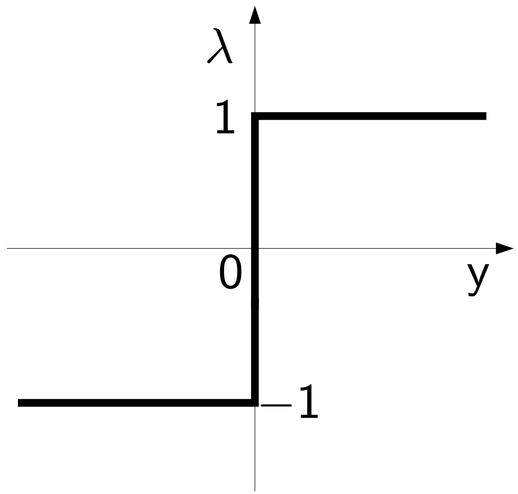
with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$, for m constraints.



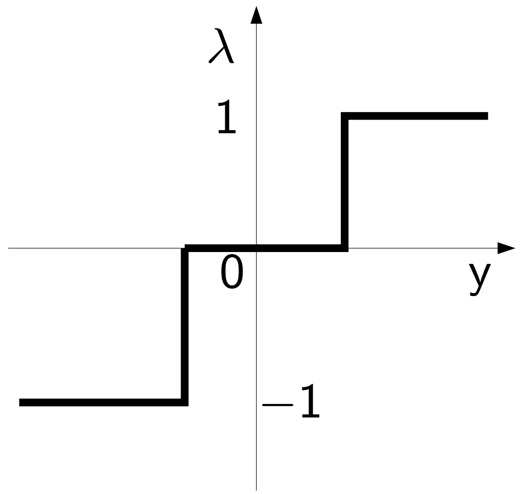
Piecewise linear systems



saturation



Relay



Relay with dead zone

Lagrangian systems with Contact and Coulomb's Friction

- Lagrangian dynamical system :

$$M(q)\ddot{q} + Q(\dot{q}, q) + F(\dot{q}, q, t) = F_{ext}(t) + R$$

- $q \in \mathbb{R}^n$: generalized coordinates vector.
- $M \in \mathbb{R}^{n \times n}$: the inertia matrix
- $Q(\dot{q}, q)$: The non linear inertial term (Coriolis)
- $F(\dot{q}, q, t)$: the internal forces
- $F_{ext}(t) : \mathbb{R} \mapsto \mathbb{R}^n$: given external load,
- $R \in \mathbb{R}^n$ is the force due the non smooth law.

Lagrangian systems with Contact and Coulomb's Friction

- Lagrangian dynamical system :

$$M(q)\ddot{q} + Q(\dot{q}, q) + F(\dot{q}, q, t) = F_{ext}(t) + R$$

  - $q \in \mathbb{R}^n$: generalized coordinates vector.
  - $M \in \mathbb{R}^{n\times n}$: the inertia matrix
  - $Q(\dot{q}, q)$: The non linear inertial term (Coriolis)
  - $F(\dot{q}, q, t)$: the internal forces
  - $F_{ext}(t) : \mathbb{R} \mapsto \mathbb{R}^n$: given external load,
  - $R \in \mathbb{R}^n$ is the force due the non smooth law.
- Linear relations.
  - Kinematic laws from the generalized coordinates to the local coordinates at contact.

$$y = H^T q + b, \dot{y} = H^T \dot{q}$$

    Mapping H : Restriction mapping composed with a change of frame
  - By duality,

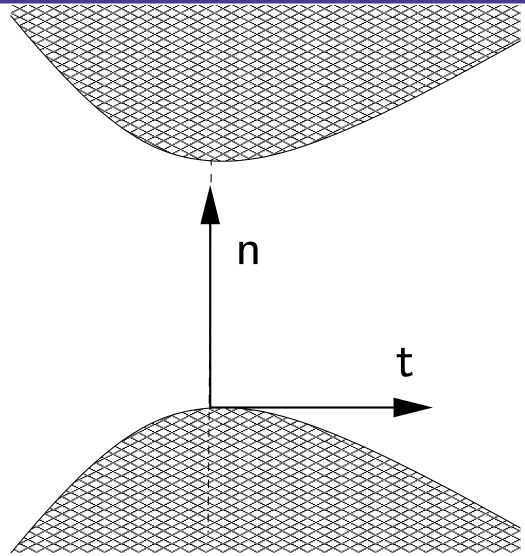
$$R = H\lambda$$

Lagrangian systems with Contact and Coulomb's Friction

- Local frame at contact : $(\boldsymbol{n}, \boldsymbol{t})$

$$y = y_{\boldsymbol{n}}\boldsymbol{n} + y_{\boldsymbol{t}}, \quad \dot{y} = \dot{y}_{\boldsymbol{n}}\boldsymbol{n} + \dot{y}_{\boldsymbol{t}}$$

$$\lambda = \lambda_{\boldsymbol{n}}\boldsymbol{n} + \lambda_{\boldsymbol{t}},$$

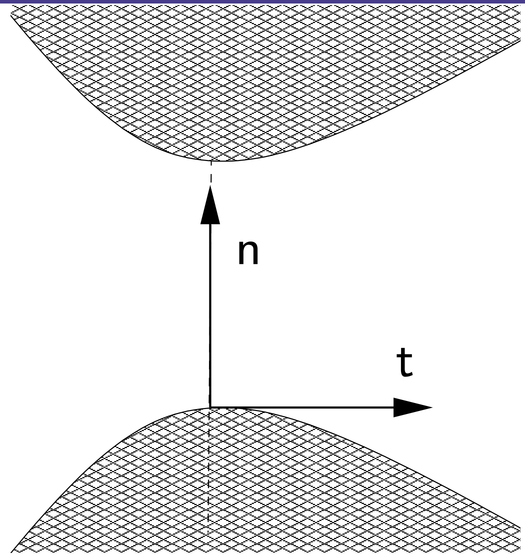


Lagrangian systems with Contact and Coulomb's Friction

- Local frame at contact : $(\boldsymbol{n}, \boldsymbol{t})$

$$y = y_{\boldsymbol{n}}\boldsymbol{n} + y_{\boldsymbol{t}}, \quad \dot{y} = \dot{y}_{\boldsymbol{n}}\boldsymbol{n} + \dot{y}_{\boldsymbol{t}}$$

$$\lambda = \lambda_{\boldsymbol{n}}\boldsymbol{n} + \lambda_{\boldsymbol{t}},$$



- Unilateral contact :

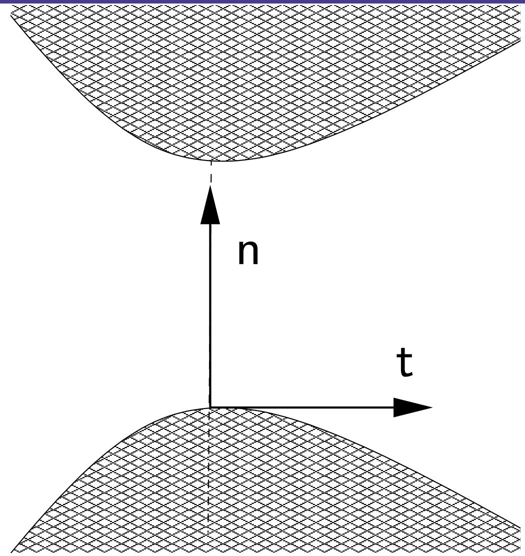
$$0 \leq y_{\boldsymbol{n}} \perp \lambda_{\boldsymbol{n}} \geq 0 \quad \Longleftrightarrow \quad -\lambda_{\boldsymbol{n}} \in \partial\Psi_{\mathbb{R}^+}(y_{\boldsymbol{n}})$$

Lagrangian systems with Contact and Coulomb's Friction

- Local frame at contact : $(\boldsymbol{n}, \boldsymbol{t})$

$$y = y_{\boldsymbol{n}}\boldsymbol{n} + y_{\boldsymbol{t}}, \quad \dot{y} = \dot{y}_{\boldsymbol{n}}\boldsymbol{n} + \dot{y}_{\boldsymbol{t}}$$

$$\lambda = \lambda_{\boldsymbol{n}}\boldsymbol{n} + \lambda_{\boldsymbol{t}},$$



- Unilateral contact :

$$0 \leq y_{\boldsymbol{n}} \perp \lambda_{\boldsymbol{n}} \geq 0 \quad \Longleftrightarrow \quad -\lambda_{\boldsymbol{n}} \in \partial\Psi_{\mathbb{R}^+}(y_{\boldsymbol{n}})$$

- Coulomb's Friction, μ Coefficient of friction, $\mathcal{C}(\mu\lambda_{\boldsymbol{n}}) = \{\lambda_{\boldsymbol{t}}, \|\lambda_{\boldsymbol{t}}\| \leq \mu\lambda_{\boldsymbol{n}}\}$

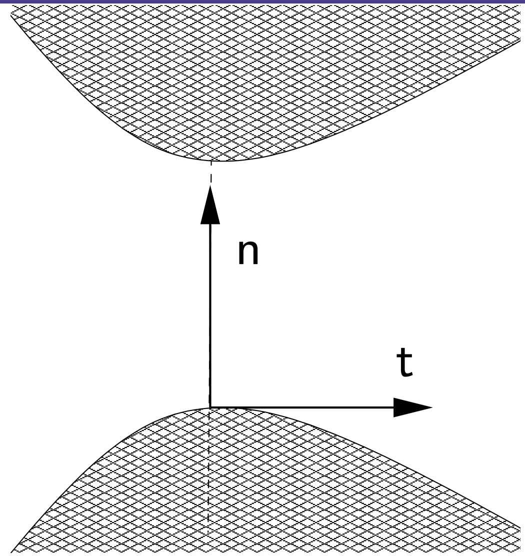
$$\begin{cases} \dot{y}_{\boldsymbol{t}} = 0, \|\lambda_{\boldsymbol{t}}\| \leq \mu\lambda_{\boldsymbol{n}} \\ \dot{y}_{\boldsymbol{t}} \neq 0, \lambda_{\boldsymbol{t}} = -\mu\lambda_{\boldsymbol{n}}\text{sign}(\dot{y}_{\boldsymbol{t}}) \end{cases} \Longleftrightarrow \dot{y}_{\boldsymbol{t}} \in \partial\Psi_{\mathcal{C}(\mu\lambda_{\boldsymbol{n}})}(-\lambda_{\boldsymbol{t}}) \Longleftrightarrow -\lambda_{\boldsymbol{t}} \in \partial\Psi^*_{\mathcal{C}(\mu\lambda_{\boldsymbol{n}})}(\dot{y}_{\boldsymbol{t}})$$

Lagrangian systems with Contact and Coulomb's Friction

- Local frame at contact : $(\boldsymbol{n}, \boldsymbol{t})$

$$y = y_{\boldsymbol{n}}\boldsymbol{n} + y_{\boldsymbol{t}}, \quad \dot{y} = \dot{y}_{\boldsymbol{n}}\boldsymbol{n} + \dot{y}_{\boldsymbol{t}}$$

$$\lambda = \lambda_{\boldsymbol{n}}\boldsymbol{n} + \lambda_{\boldsymbol{t}},$$



- Unilateral contact :

$$0 \leq y_{\boldsymbol{n}} \perp \lambda_{\boldsymbol{n}} \geq 0 \quad \Longleftrightarrow \quad -\lambda_{\boldsymbol{n}} \in \partial\Psi_{\mathbb{R}^+}(y_{\boldsymbol{n}})$$

- Coulomb's Friction, μ Coefficient of friction, $\mathcal{C}(\mu\lambda_{\boldsymbol{n}}) = \{\lambda_{\boldsymbol{t}}, \|\lambda_{\boldsymbol{t}}\| \leq \mu\lambda_{\boldsymbol{n}}\}$

$$\begin{cases} \dot{y}_{\boldsymbol{t}} = 0, \|\lambda_{\boldsymbol{t}}\| \leq \mu\lambda_{\boldsymbol{n}} \\ \dot{y}_{\boldsymbol{t}} \neq 0, \lambda_{\boldsymbol{t}} = -\mu\lambda_{\boldsymbol{n}}\text{sign}(\dot{y}_{\boldsymbol{t}}) \end{cases} \Longleftrightarrow \dot{y}_{\boldsymbol{t}} \in \partial\Psi_{\mathcal{C}(\mu\lambda_{\boldsymbol{n}})}(-\lambda_{\boldsymbol{t}}) \Longleftrightarrow -\lambda_{\boldsymbol{t}} \in \partial\Psi^*_{\mathcal{C}(\mu\lambda_{\boldsymbol{n}})}(\dot{y}_{\boldsymbol{t}})$$

- (Newton) Impact law, if necessary, e coefficient of restitution

$$\dot{y}_{\boldsymbol{n}}(t^+) = -e\dot{y}_{\boldsymbol{n}}(t^-)$$

Optimal Control with state constraints

- Quadratic optimal control Problem

$$
\begin{aligned}
\min_u I(u) &= \frac{1}{2}\int_0^T (x^T Q x + u^T R u)\, dt + \frac{1}{2} x^T(T) F x(T) \\
(s.t.) \quad \dot{x}(t) &= Ax(t) + Bu(t) \\
x(0) &= x_0, \quad x(T) = x_T \\
w(t) &= Cx(t) + D \geq 0
\end{aligned}
$$

Optimal Control with state constraints

- Quadratic optimal control Problem

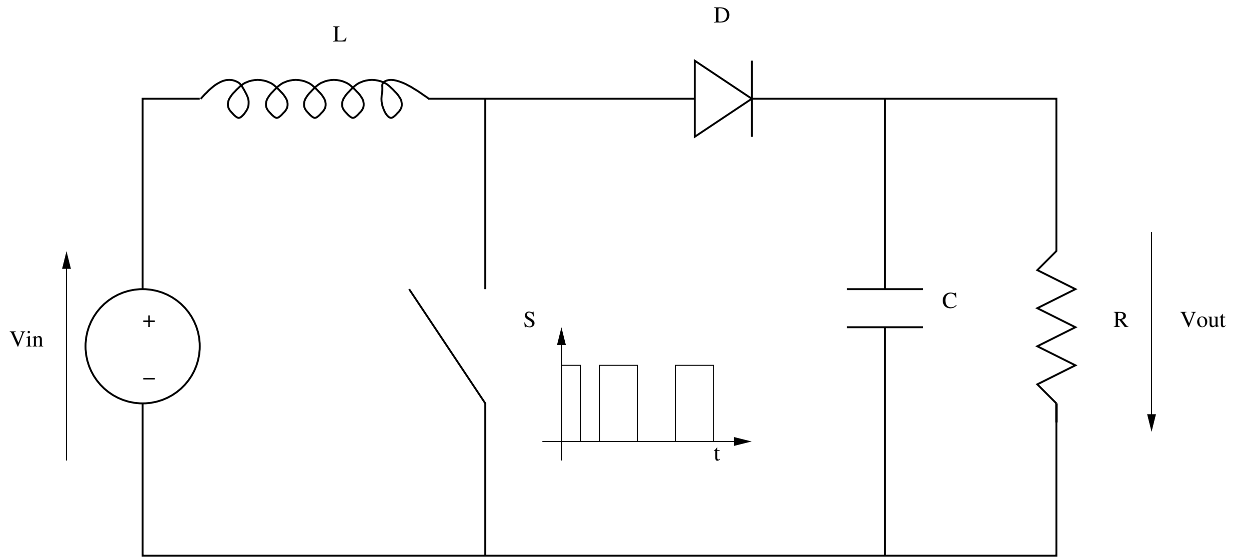
$$\begin{array}{rcl} \min\limits_{u} I(u) & = & \frac{1}{2}\int_0^T (x^T Q x + u^T R u)\, dt + \frac{1}{2} x^T(T) F x(T) \\ (s.t.) \quad \dot{x}(t) & = & Ax(t) + Bu(t) \\ x(0) & = & x_0, \quad x(T) = x_T \\ w(t) & = & Cx(t) + D \geq 0 \end{array}$$

- Necessary conditions $\Longrightarrow$ LCS with Boundary conditions :

$$\begin{array}{rcl} \begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} & = & \begin{bmatrix} A & BR^{-1}B^T \\ Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} 0 \\ -C^T \end{bmatrix} \lambda \\ 0 & \leq & Cx(t) + D \perp \lambda \geq 0 \\ x(0) & = & x_0, \quad x(T) = x_T \\ \eta(0) & = & \eta_0, \quad \eta(T) = Fx(T) + C^T\gamma + \beta = \eta_T \end{array}$$

Applications

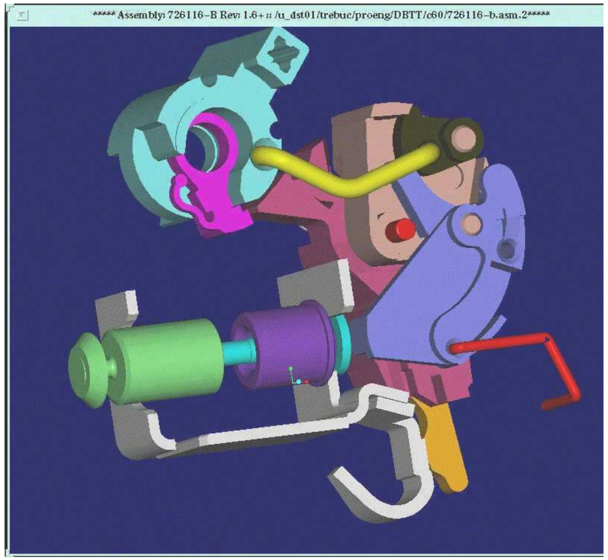
- Simulation, modeling and control of electrical networks with idealized components (diodes, transistors, switch, ...)



DC-DC Boost Converter with Sliding mode control

Applications

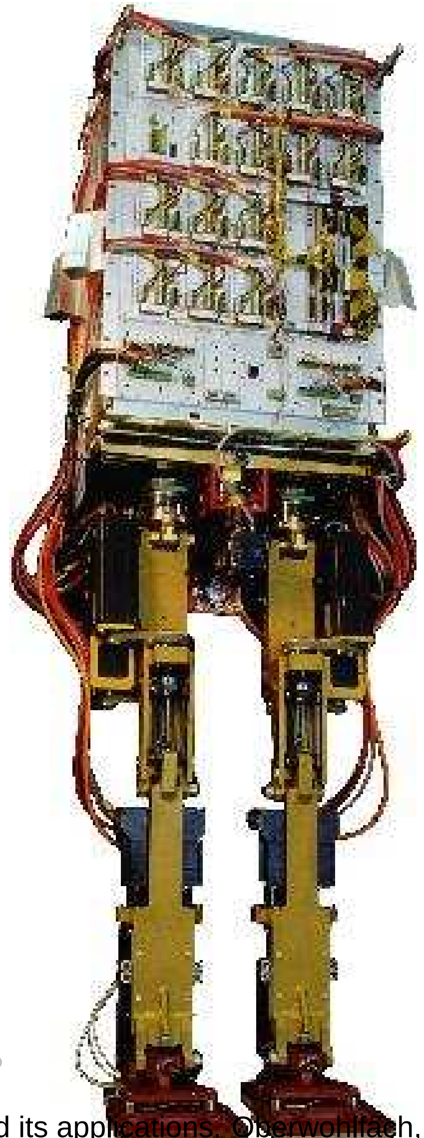
- Simulation, modeling and control of electrical networks with idealized components (diodes, transistors, switch, ...)
- Simulation, modeling and control of mechanical systems
  Simulation of Circuit breakers (INRIA/Schneider Electric)



Applications

- Simulation, modeling and control of electrical networks with idealized components (diodes, transistors, switch, ...)
- Simulation, modeling and control of mechanical systems

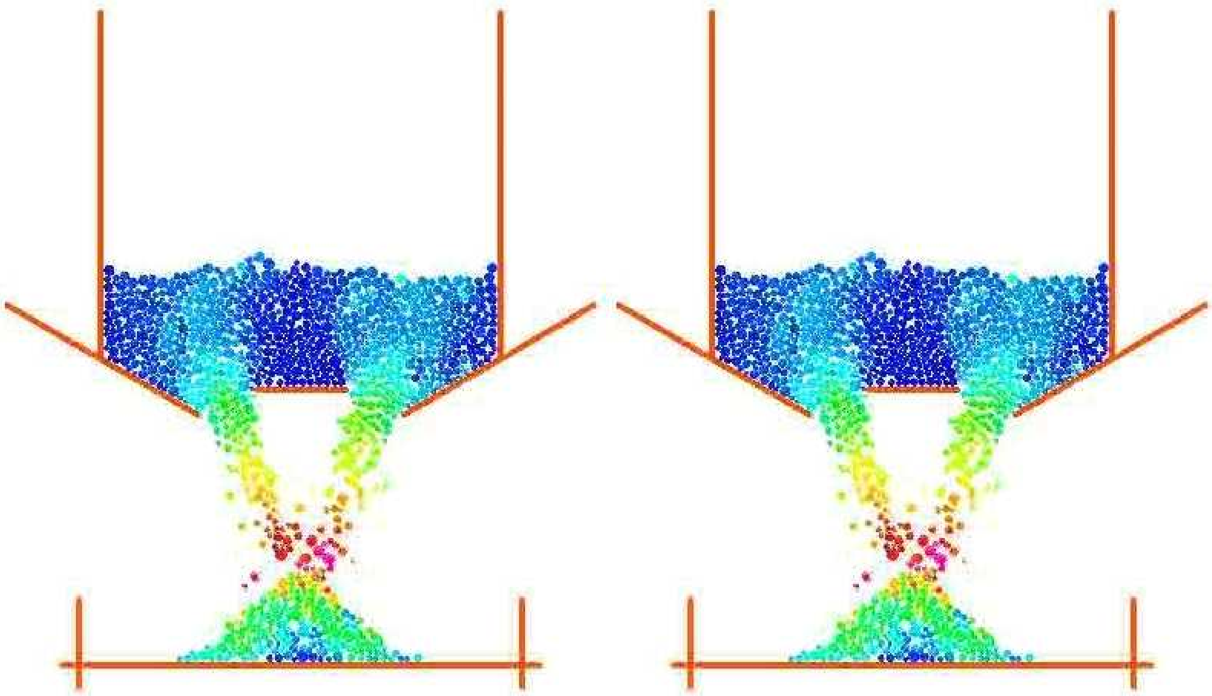
Bipedal Robot INRIA BIPOP



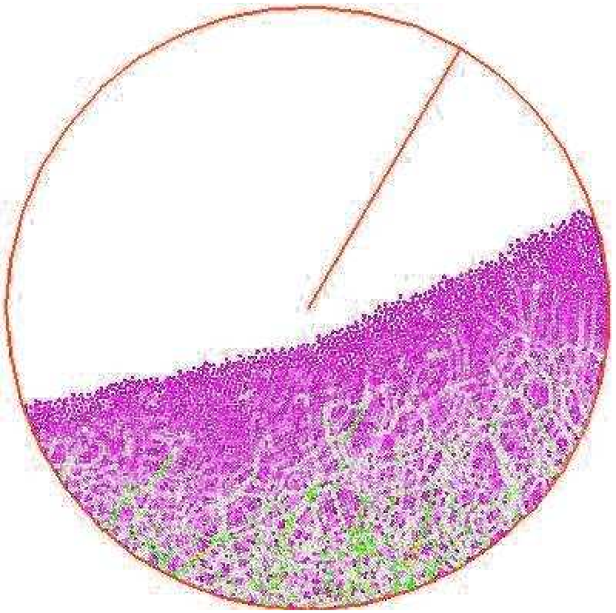
Applications

- Simulation, modeling and control of electrical networks with idealized components (diodes, transistors, switch, ...)
- Simulation, modeling and control of mechanical systems

Granular flow in a silo
LMGC Montpellier



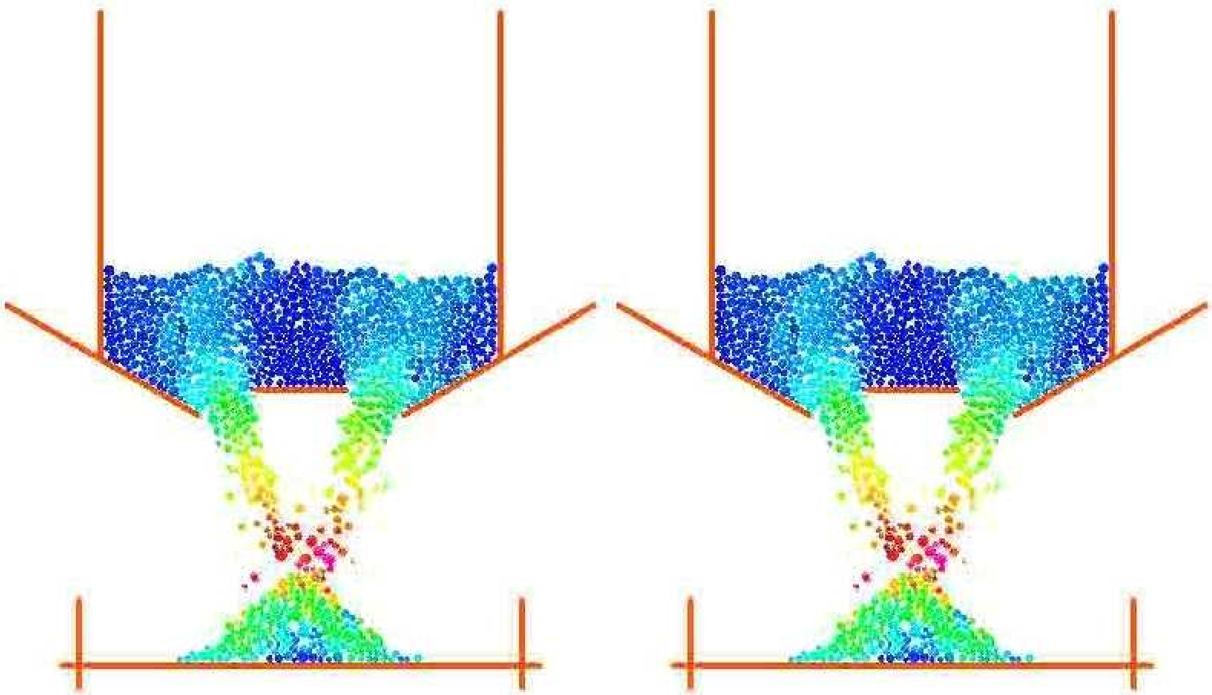
Granular Segregation
LMGC Montpellier



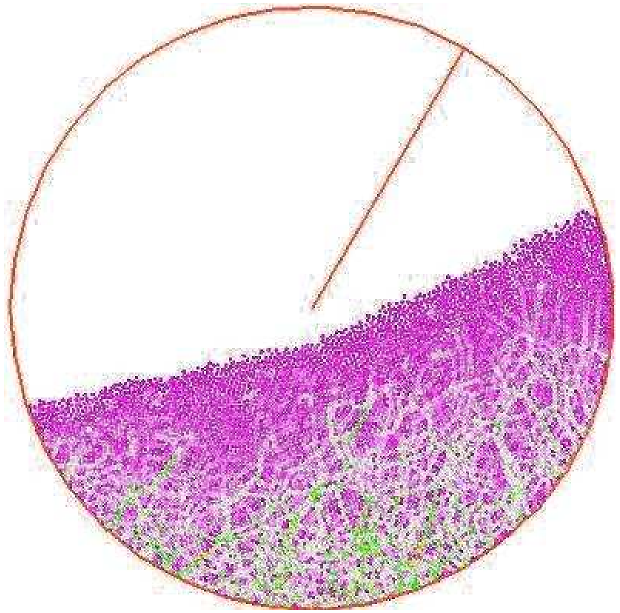
Applications

- Simulation, modeling and control of electrical networks with idealized components (diodes, transistors, switch, ...)
- Simulation, modeling and control of mechanical systems

Granular flow in a silo
LMGC Montpellier



Granular Segregation
LMGC Montpellier



- There are also applications in biology, macro-economics, ..

Outline

Difficulties and Approaches

Two major difficulties :

- Time integration of non smooth evolution
- Solving a optimization problem together with a dynamical equilibrium constraint

Two major approaches :

- Hybrid multi-modal dynamical system : Event–Driven Approach
  For a set of unilateral constraints, $y_\alpha = h_\alpha(x) \geq 0, \alpha = 1 \ldots \nu$, we define the index set of active constraints as : $I = \{\alpha, y_\alpha = 0\}$ and associated modes. An Event is a change in the index set of active constraints and a change of mode
  - Advantages
    - Easy to handle from the computational point of view : smooth integration between two events (ODE/DAE). At event, a optimization problem is solved without time evolution.
  - Disadvantages :
    - Need an accurate event detection
    - Accumulation of events
    - No existence or uniqueness results
  - Lead to Numerical Event–Driven schemes suitable :
    - Small systems with a small number of events

Difficulties and Approaches (continued ...)

Unbounded Differential inclusion and Sweeping process

- Advantages
  - Compact formulation which allow existence and uniqueness results
  - Dissipativity and monotonicity properties
- Disadvantages :
  - More difficult mathematical framework
  - Low order accuracy
- Lead to Time–stepping integration schemes (without event-handling) suitable :
  - Large systems with a large number of events
  - Accumulation of events in finite time
  - Convergence results and Existence proofs

Moreau's Sweeping Process of order 1

The Moreau's Sweeping Process is a kind of unbounded differential inclusion (Moreau 1971, 1977, Brezis 1973) :

$$\begin{cases} x(0) \in K(0) \subset \mathbb{R}^n \\ \dot{x} \in \mathcal{N}_{K(t)}(x(t)) \end{cases} \iff \begin{cases} x(0) \in K(0) \subset \mathbb{R}^n \\ \dot{x} = \lambda \\ K \ni x \perp \lambda \in N_{K(t)}(x(t)) \end{cases}$$

where $K(t)$ is a convex set

Major results :

- If $K(t)$ is bounded and a Lipschitz-continuous multi-function (Hausdorff distance) then there exists a unique solution, which is Lipschitz-continuous with the respect to time.
- If $K(t)$ is a multi-function with right continuous bounded variation then there exists a unique solution, which is of bounded variation and right continuous (Monteiro–Marques, 1987)

References on seminal works :

- Moreau, J.J. (1971) *Rafle par un convexe variable*, Séminaire d'analyse convexe
- Moreau, J.J. (1977) *Evolution problem associated with a moving convex set in a Hilbert space*, J. of Differential Equation, pp. 347–374
- Brezis, H. (1973) *Maximal Monotone operators*, North–Holland Publishing.

Moreau's Sweeping Process of order 1 (Continued...)

Equivalent Formulations

Other equivalent formulations with Projected dynamical system, Differential inclusion and variational inequalities may be found in : V.Acary, B. Brogliato, C. Lemaréchal and A. Daniilidis, Inria Research Report, RR-5107 to appear Systems and Control Letters, 2005

The Time discretization is given by the *Catching up algorithm* for instance for $K = \mathbb{R}^+$:

$$\begin{cases} x_{k+1} - x_k = h\lambda_{k+1} \\ 0 \leq x_{k+1} \perp \lambda_{k+1} \geq 0 \end{cases} \tag{1}$$

- Remarks :
  - Implicit type scheme (necessary for the unilateral constraints) but low order =1
  - Resolution of a LCP at each time step
  - Convergence Proofs => Existence of solution due to the monotonicity of the operator

This algorithm may be used for LCS system for which D is P-matrix.

Moreau's Sweeping Process of order 2

- The velocity is no longer a smooth function of time but a function of bounded variations. This is the case of Lagrangian systems
- Lagrangian dynamical system is reformulated as a measure differential equation.

$$M(q)dv + (Q(v,q) + F(v,q,t))\,dt = F_{ext}(t)\,dt + R$$

where

- dt is the Lebesgue measure on $\mathbb{R}$
- dv is the Stieltjes measure (Differential measure) associated with the right continuous function $v(t)$ of bounded variations, such that :

$$dv((a,b]) = \int_{(a,b]} dv = v(b^+) - v(a^+)$$

- R is a measure due to the non smooth law
- $q(t)$ is the absolutely continuous displacement given by :

$$q(t) = q(t_0) + \int_{t_0}^{t} v(s)\,ds$$

Moreau's Sweeping Process of order 2 (Continued ...)

Reformulation of the constraints as a measure inclusion

- Reformulation of the unilateral constraints in terms of derivatives :

$$-\lambda \in \partial\Psi_{V(y)}(\dot{y})$$

where $V(y)$ is the tangent cone of K at y which can be stated equivalently for $K = \mathbb{R}^+$ as for

$$\text{If } y(t) = 0, \text{ then } 0 \leq \dot{y} \perp \lambda \geq 0 \tag{2}$$

It's noteworthy that the (switched off) constraints is now on the velocity $\dot{y}$ and depend on the value of $y(t) = 0$

Moreau's Sweeping Process of order 2 (Continued ...)

Reformulation of the constraints as a measure inclusion

- Reformulation of the unilateral constraints in terms of derivatives :

$$-\lambda \in \partial \Psi_{V(y)}(\dot{y})$$

where $V(y)$ is the tangent cone of K at y which can be stated equivalently for $K = \mathbb{R}^+$ as for

$$\text{If } y(t) = 0, \text{ then } 0 \leq \dot{y} \perp \lambda \geq 0 \quad (2)$$

It's noteworthy that the (switched off) constraints is now on the velocity $\dot{y}$ and depend on the value of $y(t) = 0$

- If λ is a measure, the inclusion is extended considering the Radon-Nykodym derivative

$$\lambda'(t) = \frac{d\lambda}{d\nu} \in \partial \Psi_{V(y)}(\dot{y})$$

where $d\nu$ is a nonnegative measure and λ is absolutely continuous with respect to $d\nu$

Moreau's Sweeping Process. Discretization

- Given a subdivision of a time interval, $\{t_0, t_1, \ldots, t_i, \ldots, t_N\}$, we evaluate of the measure differential equation on a time interval $(t_i, t_{i+1}]$ of length h :

$$M dv((t_i, t_{i+1}]) = \int_{(t_i,t_{i+1}]} M\, dv = M(v(t^+_{i+1}) - v(t^+_i)) = \int_{t_i}^{t_{i+1}} F_{ext}(t)\, dt + \int_{(t_i,t_{i+1}]} R$$

Moreau's Sweeping Process. Discretization

- Given a subdivision of a time interval, $\{t_0, t_1, \ldots, t_i, \ldots, t_N\}$, we evaluate of the measure differential equation on a time interval $(t_i, t_{i+1}]$ of length h :

$$M dv((t_i, t_{i+1}]) = \int_{(t_i,t_{i+1}]} M\, dv = M(v(t_{i+1}^+) - v(t_i^+)) = \int_{t_i}^{t_{i+1}} F_{ext}(t)\, dt + \int_{(t_i,t_{i+1}]} R$$

- Evaluation of the displacement $q(t_{i+1}) = q(t_i) + \int_{t_i}^{t_{i+1}} v(s)\, ds$

Moreau's Sweeping Process. Discretization

- Given a subdivision of a time interval, $\{t_0, t_1, \ldots, t_i, \ldots, t_N\}$, we evaluate of the measure differential equation on a time interval $(t_i, t_{i+1}]$ of length h :

$$M dv((t_i, t_{i+1}]) = \int_{(t_i,t_{i+1}]} M\, dv = M(v(t^+_{i+1}) - v(t^+_i)) = \int_{t_i}^{t_{i+1}} F_{ext}(t)\, dt + \int_{(t_i,t_{i+1}]} R$$

- Evaluation of the displacement $q(t_{i+1}) = q(t_i) + \int_{t_i}^{t_{i+1}} v(s)\, ds$
- The measure $R((t_i, t_{i+1}])$ of the time-interval $(t_i, t_{i+1}]$ is kept as primary unknown :

$$R_{i+1} = R((t_i, t_{i+1}])$$

Moreau's Sweeping Process. Discretization

- Given a subdivision of a time interval, $\{t_0, t_1, \ldots, t_i, \ldots, t_N\}$, we evaluate of the measure differential equation on a time interval $(t_i, t_{i+1}]$ of length h :

$$M dv((t_i, t_{i+1}]) = \int_{(t_i, t_{i+1}]} M\, dv = M(v(t_{i+1}^+) - v(t_i^+)) = \int_{t_i}^{t_{i+1}} F_{ext}(t)\, dt + \int_{(t_i, t_{i+1}]} R$$

- Evaluation of the displacement $q(t_{i+1}) = q(t_i) + \int_{t_i}^{t_{i+1}} v(s)\, ds$

- The measure $R((t_i, t_{i+1}])$ of the time-interval $(t_i, t_{i+1}]$ is kept as primary unknown :

$$R_{i+1} = R((t_i, t_{i+1}])$$

Interpretation : The measure R may be decomposed as follows :

$$R = R_a\, dt + R_s$$

where $R_a\, dt$ is the abs. continuous part of the measure R and R_s the singular part.

- Impulse : If $R_a = 0$ and $R_s = P\delta_{t_{i+1}}$ then $R_{i+1} = P$
- Continuous multiplier : If $R_a(t) = f(t)$ and $R_s = 0$ then $R_{i+1} = \int_{t_i}^{t_{i+1}} f(t)\, dt$

Discretization of the Dynamics Continued

- Notations :

$$v_i \approx v(t_i^+), \quad q_i \approx q(t_i)$$

Discretization of the Dynamics Continued

- Notations :

$$v_i \approx v(t_i^+), \quad q_i \approx q(t_i)$$

- Approximation of the integral of functions : θ-method

$$\int_{t_i}^{t_{i+1}} F_{ext}(t)\,dt \approx h\,[\theta F_{ext}(t_{i+1}) + (1-\theta)F_{ext}(t_i)]$$

$$q_{i+1} = q_i + h\,[\theta v_{i+1} + (1-\theta)v_i]$$

Discretization of the Dynamics Continued

- Notations :

$$v_i \approx v(t_i^+), \quad q_i \approx q(t_i)$$

- Approximation of the integral of functions : θ-method

$$\int_{t_i}^{t_{i+1}} F_{ext}(t)\,dt \approx h\left[\theta F_{ext}(t_{i+1}) + (1-\theta) F_{ext}(t_i)\right]$$

$$q_{i+1} = q_i + h\left[\theta v_{i+1} + (1-\theta) v_i\right]$$

- Complete set of discrete equations:

$$\begin{cases} M(v_{i+1} - v_i) = h\left[\theta F_{ext}(t_{i+1}) + (1-\theta)(F_{ext}(t_i)\right] + R_{i+1} \\ q_{i+1} = q_i + h\left[\theta v_{i+1} + (1-\theta) v_i\right] \end{cases}$$

Discretization of the Dynamics Continued

- Notations :

$$v_i \approx v(t_i^+), \quad q_i \approx q(t_i)$$

- Approximation of the integral of functions : θ-method

$$\int_{t_i}^{t_{i+1}} F_{ext}(t)\,dt \approx h\left[\theta F_{ext}(t_{i+1}) + (1-\theta)F_{ext}(t_i)\right]$$

$$q_{i+1} = q_i + h\left[\theta v_{i+1} + (1-\theta)v_i\right]$$

- Complete set of discrete equations:

$$\begin{cases} M(v_{i+1} - v_i) = h\left[\theta F_{ext}(t_{i+1}) + (1-\theta)(F_{ext}(t_i)\right] + R_{i+1} \\ q_{i+1} = q_i + h\left[\theta v_{i+1} + (1-\theta)v_i\right] \end{cases}$$

- One step linear system : $v_{i+1} = v_{free} + hWR_{i+1}$ with

$$W = M^{-1}, \quad v_{free} = v_i + W\left[h\left[\theta F_{ext}(t_{i+1}) + (1-\theta)F_{ext}(t_i)\right]\right]$$

Discretization of the Dynamics Continued

- Relations at t_{k+1}:

$$\begin{aligned} y_{i+1} &= H^T q_{i+1} + b \\ \dot{y}_{i+1} &= H^T v_{i+1} \\ R_{i+1} &= H \lambda_{i+1} \end{aligned}$$

Discretization of the Dynamics Continued

- Relations at t_{k+1}:

$$\begin{aligned} y_{i+1} &= H^T q_{i+1} + b \\ \dot{y}_{i+1} &= H^T v_{i+1} \\ R_{i+1} &= H\lambda_{i+1} \end{aligned}$$

- Discretization of an unilateral constraint :
  A natural way :

$$0 \leq y_{i+1} \perp \lambda_{i+1} \geq 0$$

  in terms of velocity

$$\text{If } y^p \leq 0, \text{ then } 0 \leq \dot{y}_{i+1} \perp \lambda_{i+1} \geq 0$$

  where y^p is a prediction of the position at time t_{i+1}, for instance, $y^p = y_i + \frac{h}{2}\dot{y}_i$.

Discretization of the Dynamics Continued

- Relations at t_{k+1}:

$$\begin{aligned} y_{i+1} &= H^T q_{i+1} + b \\ \dot{y}_{i+1} &= H^T v_{i+1} \\ R_{i+1} &= H\lambda_{i+1} \end{aligned}$$

- Discretization of an unilateral constraint :
  A natural way :

$$0 \leq y_{i+1} \perp \lambda_{i+1} \geq 0$$

  in terms of velocity

$$\text{If } y^p \leq 0, \text{ then } 0 \leq \dot{y}_{i+1} \perp \lambda_{i+1} \geq 0$$

  where y^p is a prediction of the position at time t_{i+1}, for instance, $y^p = y_i + \frac{h}{2}\dot{y}_i$.

- Newton Impact law $\dot{y}^e_{i+1} = \dot{y}_{i+1} + e\dot{y}_i$

Summary

One step linear problem

$$\begin{cases} v_{i+1} = v_{free} + hWR_{i+1} \\ q_{i+1} = q_i + h\,[\theta v_{i+1} + (1-\theta)v_i] \end{cases}$$

Relations

$$\begin{cases} \dot{y}_{i+1} = H^T v_{i+1} \\ R_{i+1} = H\lambda_{i+1} \end{cases}$$

Non Smooth Law

$$\begin{cases} \text{If } y^p = y_i + \frac{h}{2}\dot{y}_i \\ \text{then } 0 \leq \dot{y}^e_{i+1} \perp \lambda_{i+1} \geq 0 \end{cases}$$

Summary

One step linear problem
$$\begin{cases} v_{i+1} = v_{free} + hWR_{i+1} \\ q_{i+1} = q_i + h\left[\theta v_{i+1} + (1-\theta)v_i\right] \end{cases}$$

Relations
$$\begin{cases} \dot{y}_{i+1} = H^T v_{i+1} \\ R_{i+1} = H\lambda_{i+1} \end{cases}$$

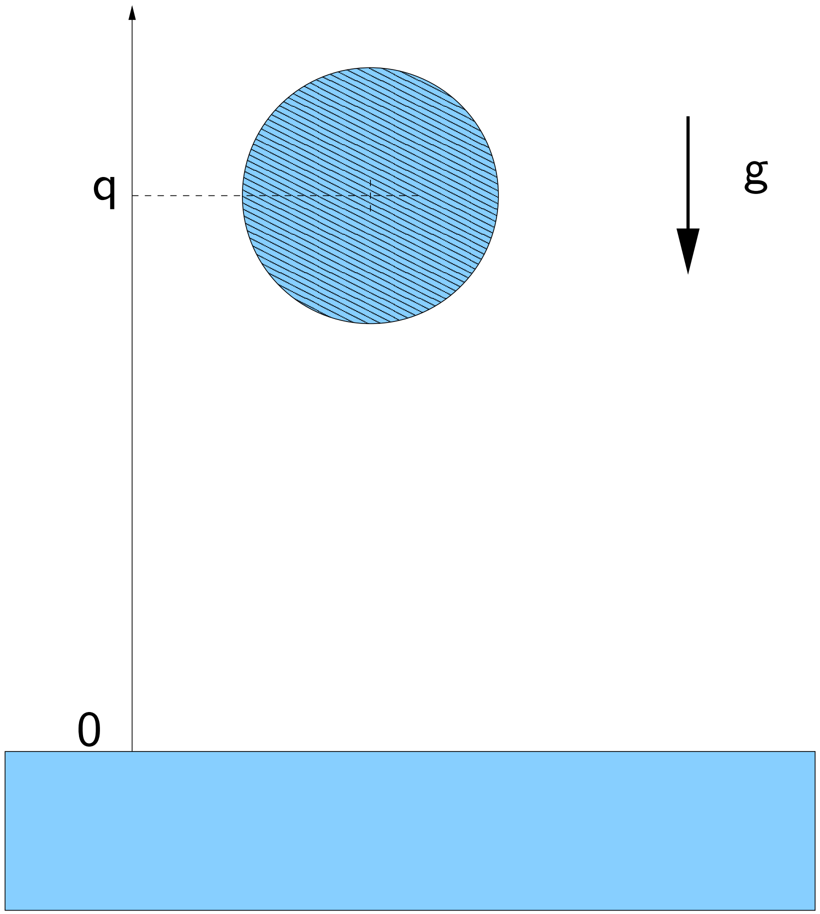
Non Smooth Law
$$\begin{cases} \text{If } y^p = y_i + \dfrac{h}{2}\dot{y}_i \\ \text{then } 0 \leq \dot{y}^e_{i+1} \perp \lambda_{i+1} \geq 0 \end{cases}$$

➜ One step Quasi-LCP in terms of $\dot{y}^e_{i+1}$ and λ_{i+1} :

$$\begin{aligned} \dot{y}^e_{i+1} &= H^T \dot{q}_{free} + hH^T WH\lambda_{i+1} + e\dot{y}_i \\ y^p &= y_i + \frac{h}{2}\dot{y}_i \\ \text{If} \quad y^p &\leq 0, \text{ then } 0 \leq \dot{y}^e_{i+1} \perp \lambda_{i+1} \geq 0 \end{aligned}$$

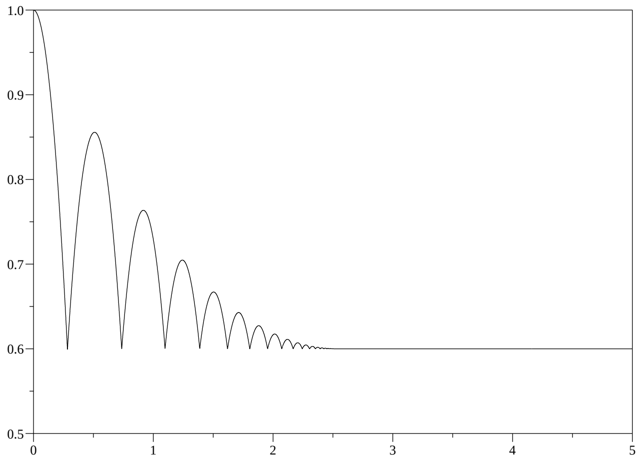
A simple example : A bouncing ball

The Bouncing ball example with time–stepping

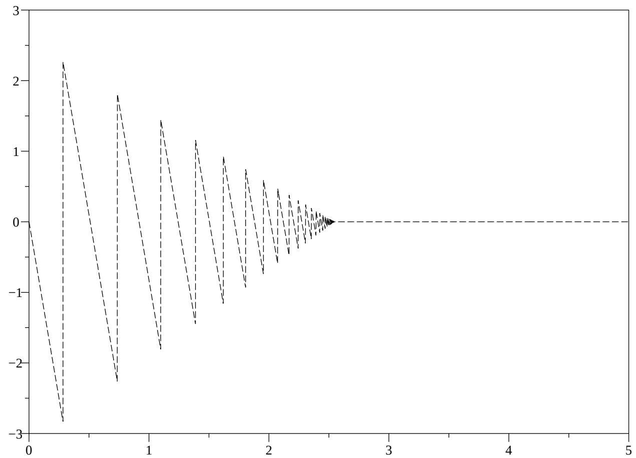


$$\begin{cases} m\ddot{q} = -mg + \lambda \\ \text{if } q(t) = 0, \\ 0 \leq \dot{q}(t^+) + e\dot{q}(t^-) \perp \lambda \geq 0 \end{cases}$$

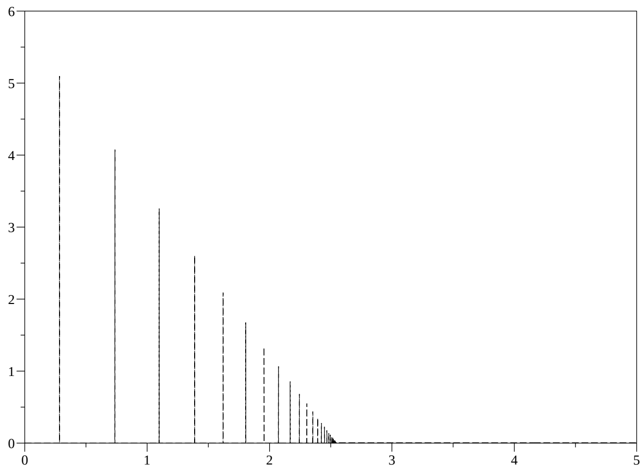
A simple example : A bouncing ball



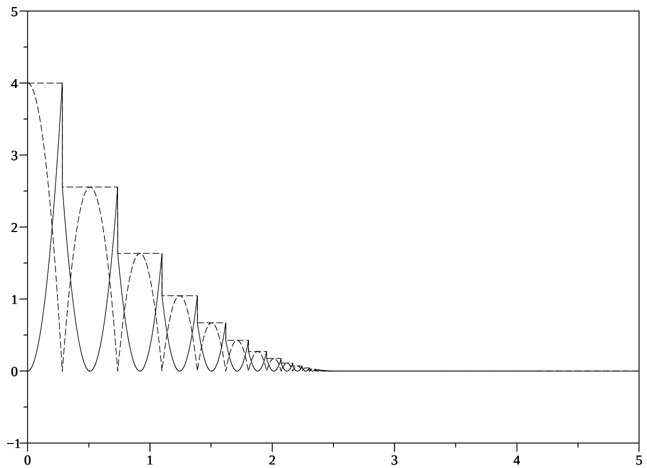
Position of the ball vs. Time



Velocity of the ball vs. Time



Reaction due to the contact force vs. Time



Energy balance vs.time

Open Problems and links with optimization

- Efficient algorithm for the LCP with a switched-off constraints :

$$\begin{array}{rcl} y & = & A\lambda + b \\ v & = & h(y) \\ \text{If } y & \leq & 0, \text{ then } \quad 0 \leq v \perp \lambda \geq 0 \end{array}$$

  - Issue ? : Reformulation in terms of QP with a additional slack variable (MIP)?

  ➜ Good BVP solvers for which a prediction of y is not reasonable

Open Problems and links with optimization

- Efficient algorithm for the LCP with a switched-off constraints :

$$\begin{array}{rcl} y &=& A\lambda + b \\ v &=& h(y) \\ \text{If} \quad y &\leq& 0, \text{then} \quad 0 \leq v \perp \lambda \geq 0 \end{array}$$

  - Issue ? : Reformulation in terms of QP with a additional slack variable (MIP)?

  ➜ Good BVP solvers for which a prediction of y is not reasonable

- Efficient algorithm for the 3D Frictional contact problem with or without switched-off constraints :

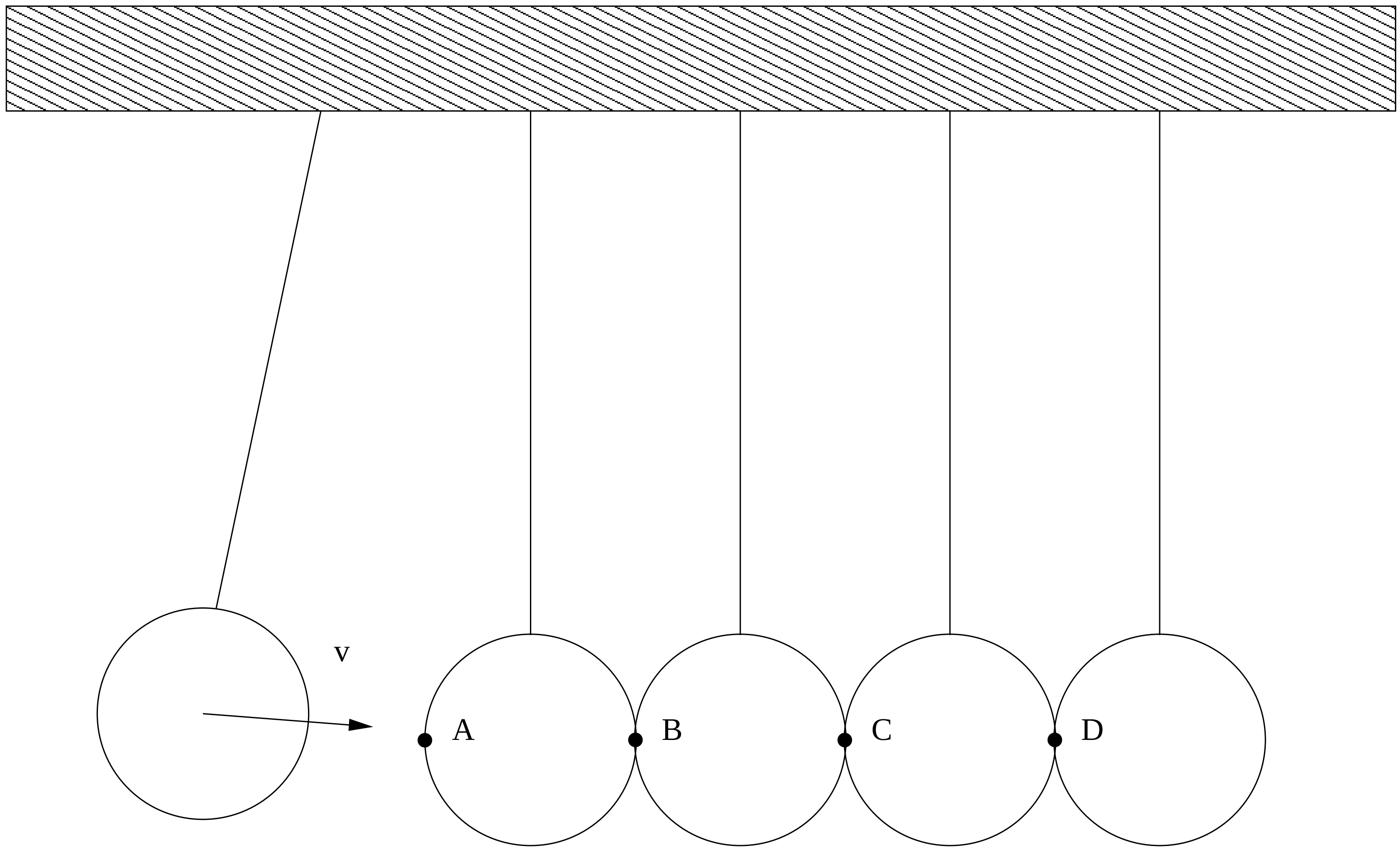
$$\begin{array}{rcl} y &=& A\lambda + b \\ y &=& [y_n, y_t], \quad \lambda = [\lambda_n, \lambda_t] \\ v &=& h(y) = [\lambda_n, \lambda_t] \\ (\text{If} \quad y_n &\leq& 0, \text{then}) \begin{cases} 0 \leq v_n \perp \lambda_n \geq 0 \\ -\lambda_t \in \partial \Psi^*_{C(\mu\lambda_n)}(v_n) \end{cases} \end{array}$$

  We use basic and robust iterative scheme (Gauss-Seidel like) and (Non smooth) Generalized Newton Method (Alart and Curnier, 1990)

  - Issue ? : - Try to find a good potential to minimize and/or a good Lagrangian relaxation ?
    - NLCP solvers ? Bundle Methods ?
    - Good line searches/trust regions for Generalized Newton Method ?

Open Problems and links with optimization (Continued)

- Energetic coefficient of restitution e and multiple impact law



Newton's Cradle

Vincent ACARY
BipOp project, INRIA Rhône–Alpes

Open Problems and links with optimization (Continued)

- Energetic coefficient of restitution e and multiple impact law
  Find $(u, v, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ giving $M \succ 0, H, \Theta, b$:

$$\begin{array}{rcll} M(u-b) & = & H\lambda & \\ u^T M u & = & e\, b^T M b^T, & \text{(Energy dissipation)} \\ v & = & H^T u \geq 0 & \\ \lambda & \geq & 0 & \\ \Theta\lambda & = & 0, & \text{(Multiple Impact Law at distance)} \end{array}$$

  We can also add friction : Find $(u, v, \lambda \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ giving $M \succ 0, H, \Theta, b$:

$$\begin{array}{rcll} M(u-b) & = & H[\lambda_n, \lambda_t]^T & \\ u^T M u & = & e\, b^T M b^T, & \text{Energy dissipation} \\ v & = & [v_n, v_t]^T = H^T u \geq 0, & \\ \lambda_n & \geq & 0 & \\ \Theta\lambda_n & = & 0 & \text{(Multiple Impact Law at distance)} \\ -\lambda_t & \in & \partial\Psi^*_{C(\mu\lambda_n)}(v_t) & \end{array}$$

Open Problems and links with optimization (Continued)

- **Energetic coefficient of restitution e and multiple impact law**
  Find $(u, v, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ giving $M \succ 0, H, \Theta, b$:

$$\begin{array}{rcl} M(u-b) & = & H\lambda \\ u^T M u & = & e\, b^T M b^T, \qquad \text{(Energy dissipation)} \\ v & = & H^T u \geq 0 \\ \lambda & \geq & 0 \\ \Theta\lambda & = & 0, \quad \text{(Multiple Impact Law at distance)} \end{array}$$

  We can also add friction : Find $(u, v, \lambda \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ giving $M \succ 0, H, \Theta, b$:

$$\begin{array}{rcl} M(u-b) & = & H[\lambda_n, \lambda_t]^T \\ u^T M u & = & e\, b^T M b^T, \qquad \text{Energy dissipation} \\ v & = & [v_n, v_t]^T = H^T u \geq 0, \\ \lambda_n & \geq & 0 \\ \Theta\lambda_n & = & 0 \quad \text{(Multiple Impact Law at distance)} \\ -\lambda_t & \in & \partial\Psi^*_{C(\mu\lambda_n)}(v_t) \end{array}$$

- **Extension to Non linear mechanical behavior $y = f(\lambda)$** We use only outer linearization with Newton-Raphson scheme

Outline

Introduction

Joint Work with :

- Bernard Brogliato, Head of the Bipop Project, INRIA Rhône-Alpes
- Daniel Goeleven, IREMIA, University of La Réunion

References :

- V. Acary and B. Brogliato, *Higher Order Moreau's sweeping process*, Colloquium in the honor of the 80th Birthday of J.J. Moreau, to appear in "Non smooth Mechanics and Analysis: theoretical and numerical advances", Kluwer, 2005
- V. Acary, B. Brogliato and D. Goeleven, *Higher Order Moreau's sweeping process: Mathematical formulation and numerical simulation*, INRIA Research Report RR-5236 , *submitted to MPA*
- J.S Pang and D. Stewart, *Differential Variational Inequalities*, preprint, *submitted to MPA*
  - Elegant Formulations of Unbounded Differential inclusion as Variational Inequalities
  - IVP and BVP
  - New proof of convergence for time-stepping scheme
  - But only for low order systems (≤ 1)

Preliminary example on LCS

- Linear complementarity system :

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \leq y \perp \lambda \geq 0 \end{cases}$$

- Let us consider the very simple example :

$$\begin{cases} \dddot{x} = \lambda, \quad x \in \mathbb{R}, \lambda \in \mathbb{R} \\ 0 \leq y = x \perp \lambda \geq 0 \end{cases} \tag{-8}$$

Naive Remarks:

- If $x(t) = 0$ and $\dot{x}(t^-) < 0, \ddot{x}(t^-) < 0, \dddot{x}(t^-) < 0$ then all of the derivatives must jump.
- If $\dot{x}$ have a jump, $\ddot{x}$ is a measure (Dirac) and $\dddot{x}$ a derivative (in the sense of distribution) of a Dirac.
- In this case, λ is also a derivative of a Dirac and then there is no sense to require that $\lambda \geq 0$

Notion of Relative degree

- **Definition :** Defining the Markov Parameters as $(D, CB, CAB, CA^2B, \ldots)$, the relative degree r is the rank of the first non zero Markov Parameter.
- **Remarks**
  - the Relative degree r is the number of differentiation of y to obtain explicitly y in function of λ.
  - Clear Analogy with the differential index in DAE ($\delta = r + 1$)

Vincent ACARY
BipOp project, INRIA Rhône–Alpes

Notion of Relative degree (Continued ...)

- Relative degree $r = 0$, $D \succ 0$, Trivial case
  - The multiplier $\lambda = \max(0, -D^{-1}Cx)$ is a Lipschitz continuous function of x
  - The numerical integration may be performed with any standard ODE solvers.

Notion of Relative degree(Continued ...)

- Relative degree $r = 0$, $D \succ 0$, Trivial case
  - The multiplier $\lambda = \max(0, -D^{-1}Cx)$ is a Lipschitz continuous function of x
  - The numerical integration may be performed with any standard ODE solvers.
- Relative degree $r = 1$, $D = 0, CB \succ 0$
  - The multiplier λ is a function of time t, not necessarily continuous, for instance, of bounded variations (BV).
  - The numerical integration have to be performed with specific solvers (Event–Driven or Moreau's Catching up algorithm)

Notion of Relative degree(Continued ...)

- Relative degree $r = 0$, $D \succ 0$, Trivial case
  - The multiplier $\lambda = \max(0, -D^{-1}Cx)$ is a Lipschitz continuous function of x
  - The numerical integration may be performed with any standard ODE solvers.
- Relative degree $r = 1$, $D = 0, CB \succ 0$
  - The multiplier λ is a function of time t, not necessarily continuous, for instance, of bounded variations (BV).
  - The numerical integration have to be performed with specific solvers (Event–Driven or Moreau's Catching up algorithm)
- Relative degree $r = 2$, $D = 0, CB = 0, CAB \succ 0$
  - The system is not self-consistent : Need a re-initialization mapping
  - The multiplier λ is a real measure.
  - Specific solvers (Event–Driven or Moreau's Time–stepping) as for Lagrangian dynamical system with constraints

Notion of Relative degree(Continued ...)

- **Relative degree $r = 0$, $D \succ 0$, Trivial case**
  - The multiplier $\lambda = \max(0, -D^{-1}Cx)$ is a Lipschitz continuous function of x
  - The numerical integration may be performed with any standard ODE solvers.
- **Relative degree $r = 1$, $D = 0, CB \succ 0$**
  - The multiplier λ is a function of time t, not necessarily continuous, for instance, of bounded variations (BV).
  - The numerical integration have to be performed with specific solvers (Event–Driven or Moreau's Catching up algorithm)
- **Relative degree $r = 2$, $D = 0, CB = 0, CAB \succ 0$**
  - The system is not self-consistent : Need a re-initialization mapping
  - The multiplier λ is a real measure.
  - Specific solvers (Event–Driven or Moreau's Time–stepping) as for Lagrangian dynamical system with constraints
- **Higher Relative degree $r \geq 3$ $D = 0, CB = 0, CA^{r-2} = 0, \ldots, CA^{r-1}B \succ 0$**
  - The multiplier λ is a distribution of order $r - 1$.
  - Dedicated time-stepping scheme and nested complementarity problems

Issues to be fixed

- Reformulation of the problem as :
  - Canonical form (Zero-Dynamics)
  - Distributional dynamical systems
  - Measure differential equations (also possibly Measure variational Inequalities)
  - “Good” Reinitialization mapping (Monotone mapping)
- Characterization of solutions
- Mathematical results Existence and uniqueness
- Time–stepping scheme for IVP and BVP
- Efficient Algorithm for Nested Complementarity Problems

Assumptions :

- Autonomous and linear time invariant systems
- Homogeneous relative degree

Canonical form: The Zero Dynamical form (ZD)

Let us consider the following LTI system :

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ w = Cx + D\lambda \end{cases}$$

We perform a state-space transformation $z = Wx, z^T = (w, \dot{w}, \ldots, w^{(r-1)}, \xi)$ such that :

$$\begin{cases} \dot{z}_1(t) = z_2(t) \quad (t \geq 0) \\ \dot{z}_2(t) = z_3(t) \quad (t \geq 0) \\ \dot{z}_3(t) = z_4(t) \quad (t \geq 0) \\ \vdots \\ \dot{z}_{r-1}(t) = z_r(t) \quad (t \geq 0) \\ \dot{z}_r(t) = CA^r W^{-1} z(t) + CA^{r-1} B\lambda(t) \quad (t \geq 0) \\ \dot{\xi}(t) = A_\xi \xi(t) + B_\xi z_1(t) \quad (t \geq 0) \\ w(t) = z_1(t) \quad (t \geq 0) \end{cases}$$

This transformation always exists for controllable systems

Distributional Dynamics

Let us consider a system of equality distributions of Class, $\cup_{n\in\mathbb{N}}\mathcal{T}_n(I)$,

$$\left\{\begin{array}{l} Dz_1 = z_2 \\ Dz_2 = z_3 \\ Dz_3 = z_4 \\ \vdots \\ Dz_{r-1} = z_r \\ Dz_r = CA^rW^{-1}z + CA^{r-1}B\lambda \\ D\xi = A_\xi\xi + B_\xi z_1 . \end{array}\right. \iff \left\{\begin{array}{l} Dz_1 = \{z_2\} + \nu_1 \\ Dz_2 = \{z_3\} + D\nu_1 + \nu_2 \\ Dz_3 = \{z_4\} + D^2\nu_1 + D\nu_2 + \nu_3 \\ \vdots \\ Dz_i = \{z_{i+1}\} + D^{(i-1)}\nu_1 + D^{(i-2)}\nu_2 + \ldots + D\nu_{i-1} + \nu_i \\ \vdots \\ Dz_{r-1} = \{z_r\} + D^{(r-2)}\nu_1 + \ldots + \nu_{r-1} \\ Dz_r = CA^rW^{-1}\{z\} + CA^{r-1}B\lambda. \\ D\xi = A_\xi\xi + B_\xi z_1 . \end{array}\right. \quad (-7)$$

where ν_i the measure part of the distribution Dz_i

This now possible to give a meaning to the positivity of λ :

$$\lambda = (CA^{r-1}B)^{-1}[D^{(r-1)}\nu_1 + \ldots + D\nu_{r-1}] + \nu_r$$

by imposing some constraints of positivity to ν_i

Measure differential dynamics

Stronger Assumption("weaker" formalism) : requiring that the solutions z_i of the distributional dynamics are regular distributions z_i generated by right continuous functions of special locally bounded variation.

More precisely, $\xi_1, ..., \xi_{n-r} \in \mathcal{F}_\infty(\mathbb{R}^+; \mathbb{R})$ such that

$$
\begin{cases}
dz_1 = z_2(t)dt + d\nu_1 \\
dz_2 = z_3(t)dt + d\nu_2 \\
dz_3 = z_4(t)dt + d\nu_3 \\
\vdots \\
dz_i = z_{i+1}(t)dt + d\nu_i \\
\vdots \\
dz_{r-1} = z_r(t)dt + d\nu_{r-1} \\
dz_r = CA^rW^{-1}z(t)dt + CA^{r-1}Bd\nu_r \\
\dot{\xi}(t) = A_\xi \xi(t) + B_\xi z_1(t)
\end{cases}
$$

Reinitialization mapping

- **Definition of tangent cone to Φ :** Let Φ be a nonempty closed convex subset of $\mathbb{R}$. We denote by $T_\Phi(x)$ the tangent cone of Φ at $x \in \mathbb{R}$ defined by

$$T_\Phi(x) = \overline{\text{cone}}(\Phi - \{x\}) \tag{-6}$$

where $\text{cone}(\Phi - \{x\})$ denotes the cone generated by $\Phi - \{x\}$. This definition allows us to take into account constraints violations. Note that

$$T_{\mathbb{R}^+}(x) = \begin{cases} \mathbb{R} & \text{if} \quad x > 0 \\ \mathbb{R}^+ & \text{if} \quad x \leq 0 \end{cases} \text{ and } T_{\mathbb{R}}(x) = \mathbb{R}.$$

- **Definition of nested tangent cones :** Let us now set $\Phi := \mathbb{R}^+$. For $z \in \mathbb{R}^r$, we set $Z_i = (z_1, z_2, \ldots, z_i), \quad (1 \leq i \leq r)$. We define

$$T^0_\Phi(Z_1) = \Phi, \;\; T^1_\Phi(Z_1) = T_\Phi(z_1), \;\; T^2_\Phi(Z_2) = T_{T^1_\Phi(Z_1)}(z_2), \ldots T^i_\Phi(Z_i) = T_{T^{i-1}_\Phi(Z_{i-1})}(z_i).$$

- **Definition of the Reinitialization mapping :**

$$d\nu_i \in -\partial\psi_{T^{i-1}_\Phi(\{Z_{i-1}\}(t^-))}(\{z_i\}(t^+)) \quad \text{on } \tilde{I}, \quad (1 \leq i \leq r) \tag{-5}$$

Reinitialization mapping Continued ...

- Interpretation of this inclusion

If $T_{\Phi}^{i-1}(\{Z_{i-1}\}(t^-)) = \mathbb{R}^+$, i.e, if $z_1 \leq 0, z_2 \leq 0, ..., z_i \leq 0$ then one gets a complementarity condition :

$$0 \leq d\nu_i \perp \{z_i\}(t^+) \geq 0$$

otherwise

$$d\nu_i = 0$$

➜ we obtain a set of nested complementarity conditions (Generalization of $r = 2$) :

$$\begin{array}{c} 0 \leq d\nu_1 \perp \{z_1\}(t^+) \geq 0 \\ \text{if } z_1 \leq 0 \text{ then } 0 \leq d\nu_2 \perp \{z_2\}(t^+) \geq 0 \\ \text{if } z_1 \leq 0 \text{ and } z_2 \leq 0 \text{ then } 0 \leq d\nu_3 \perp \{z_3\}(t^+) \geq 0 \\ \text{if } z_1 \leq 0 \text{ and } z_2 \leq 0 \text{ and } z_3 \leq 0 \text{ then } 0 \leq d\nu_4 \perp \{z_4\}(t^+) \geq 0 \\ \vdots \end{array}$$

Well posedness results

- Definition of Regular solution :
  Let $0 \leq a < b \in \mathbb{R} \cup \{+\infty\}$ be given. We say that a solution $z \in (\mathcal{T}_{r-1}(\mathbb{R}^+))^n$ of Measure differential Inclusions is regular on $[a, b)$ if for each $t \in [a, b)$, there exists a right neighborhood $[t, \sigma) \quad (\sigma > 0)$ such that the restriction of $\{z\}$ to $[t, \sigma)$ is analytic.

- Global Existence and Uniqueness of a Regular Solution
  Suppose that $CA^{r-1}B \succ 0$. For each $z_0 \in \mathbb{R}^n$, the system of Measure differential Inclusions has at least one regular solution.

  Moreover:

  i) $z_1 \equiv \{z_1\} \geq 0$ on $\mathbb{R}^+$

  ii) $\{\bar{z}\}(0^+) = \bar{z}'_0$

  iii) $\|\{z\}(t)\| \leq e^{\|WAW^{-1}\|t}\|z_0\|, \quad \forall\, t \in \mathbb{R}^+$

  iv) If z^1 and z^2 are two regular solutions then $\langle z^1, \varphi\rangle = \langle z^2, \varphi\rangle, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^+; \mathbb{R}^n)$.

Time-discretization

- Summary of the Measure Differential inclusion :

$$\begin{cases} dz_i - z_{i+1}(t)dt = d\nu_i, 1 \leq i \leq r-1 \\ dz_r - CA^rW^{-1}z(t)dt = (CA^{r-1}B)^{-1}d\nu_r \\ d\nu_i \in -\partial\psi_{T_\Phi^{i-1}(z_1(t^-),\ldots,z_{i-1}(t^-))}(z_i(t^+)) \\ \dot{\xi}(t)\,dt = A_\xi\xi(t) + B_\xi z_1(t)\,dt \end{cases}$$

- Time discretization :
  We denote by $0 = t_0 < t_1 < \ldots < t_k < t_N = T$ a finite partition (or a subdivision) of the time interval $[0,T], T > 0$ and the time step is $h = t_{k+1} - t_k$
  The values of the measures $dz_i((t_k,t_{k+1}])$ and $\mu_{i,k+1} = d\nu_i((t_k,t_{k+1}])$ are kept as primary variables and this fact is crucial for the consistency of the method for the non smooth evolutions.

$$\begin{cases} z_{i,k+1} - z_{i,k} - hz_{i+1,k+1} = \mu_{i,k+1}. \\ z_{r,k+1} - z_{r,k} - hCA^rW^{-1}z_{k+1} = CA^{r-1}B\;\mu_{r,k+1} \\ \mu_{i,k+1} \in -\partial\psi_{T_\Phi^{i-1}(z_{1,k},\ldots,z_{i-1,k})}(z_{i,k+1}) \\ \xi_{k+1} - \xi_k = hA_\xi\xi_{k+1} + hB_\xi z_{1,k+1} \end{cases}$$

Properties of the Time-discretization

- Proposition 1: Boundedness of the sequences of approximation (z_k, ξ_k) :

$$\|z_n\| \leq \alpha, \quad \||\mu_k\|| \leq M$$

- Proposition 2: Local Bounded Variation of step function $Z_i(t)$ generated by the approximation z_i on a interval $[0,T]$

$$\mathrm{var}(z_i^N,[0,T]) \leq \frac{1}{2R}(|z_{i,0}-a|+h\alpha)^2 + \frac{\alpha^2}{2R}T^2 + \alpha T(1+\frac{1}{R}|z_{i,0}-a|) \qquad \text{for all } 1 \leq i \leq r-1$$

$$\mathrm{var}(z_r^N,[0,T]) \leq \frac{1}{2R}(|z_{r,0}-a|+h\beta\alpha)^2 + \frac{\beta^2\alpha^2}{2R}T^2 + \beta\alpha T(1+\frac{1}{R}|z_{1,0}-a|)$$

$$\mathrm{var}(\xi^N,[0,T]) \leq (\gamma+\delta)\alpha T$$

➜ Helly's Theorem : There is a subsequence of (z_k, ξ_k) that converges point-wisely towards to some function $z(t), \xi(t)$ which is of Local Bounded variations

- Still to be done:
  - Prove that this limit is a solution of Measure differential inclusion
  - Choose and define a correct topology to measure convergence between two filled in graphs of BV functions. (Hausdorff distance)
  - After that, the convergence of the scheme and the order are straightforward corollaries due to the existence and uniqueness properties of the problem

Outline

A simple example

A simple example with a non trivial zero-dynamics:

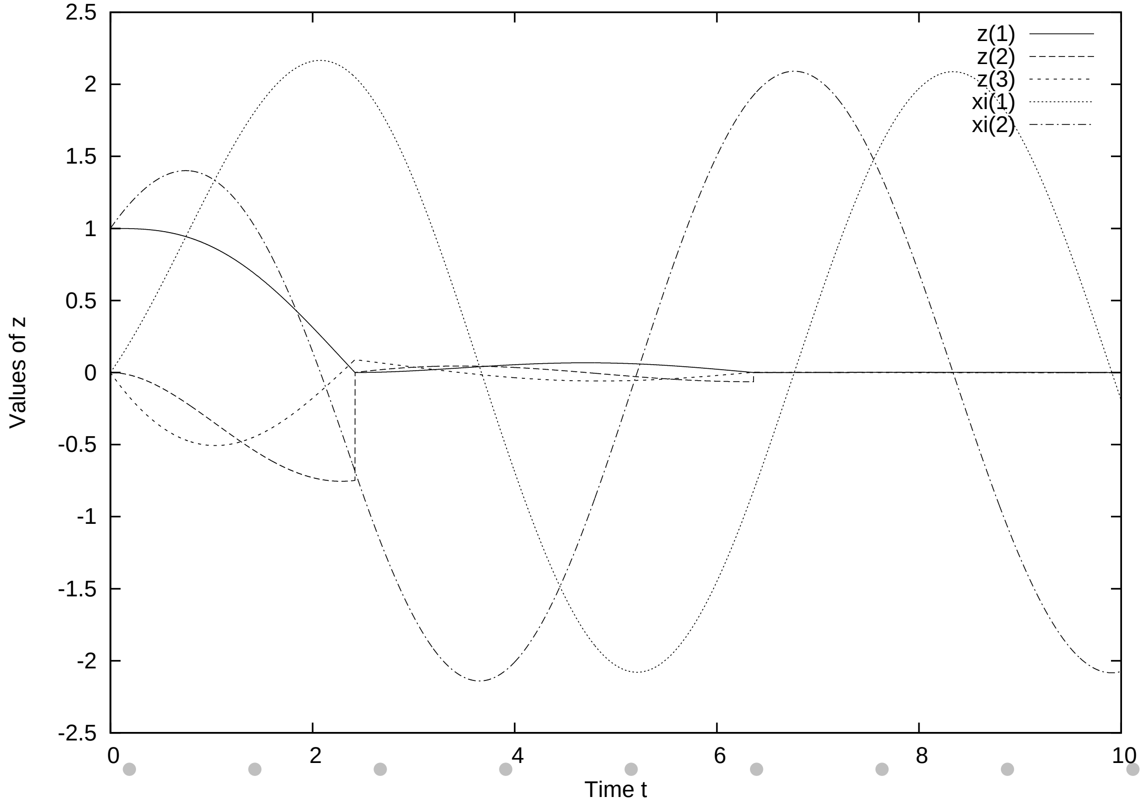
$$
\begin{cases}
z(0) = (1,0,0,0,0)^T \\
\dot{z}_1(t) = z_2(t) \\
\dot{z}_2(t) = z_3(t) \\
\dot{z}_3(t) = -z_1(t) - z_2(t) - z_3(t) - d_\xi^T \xi(t) + \lambda(t) \\
\dot{\xi}_1(t) = \alpha \xi_2(t) \\
\dot{\xi}_2(t) = -\omega \xi_1(t) + z_1(t) \\
w(t) = z_1(t) \geq 0
\end{cases}
$$

A simple example

A simple example with a non trivial zero-dynamics:

$$\begin{cases} z(0) = (1,0,0,0,0)^T \\ \dot z_1(t) = z_2(t) \\ \dot z_2(t) = z_3(t) \\ \dot z_3(t) = -z_1(t) - z_2(t) - z_3(t) - d_\xi^T \xi(t) + \lambda(t) \\ \dot \xi_1(t) = \alpha \xi_2(t) \\ \dot \xi_2(t) = -\omega \xi_1(t) + z_1(t) \\ w(t) = z_1(t) \geq 0 \end{cases}$$

For $d_\xi = (0,0)$, the zero dynamic does not play any role $\alpha = 1$ and $\omega = 1$.

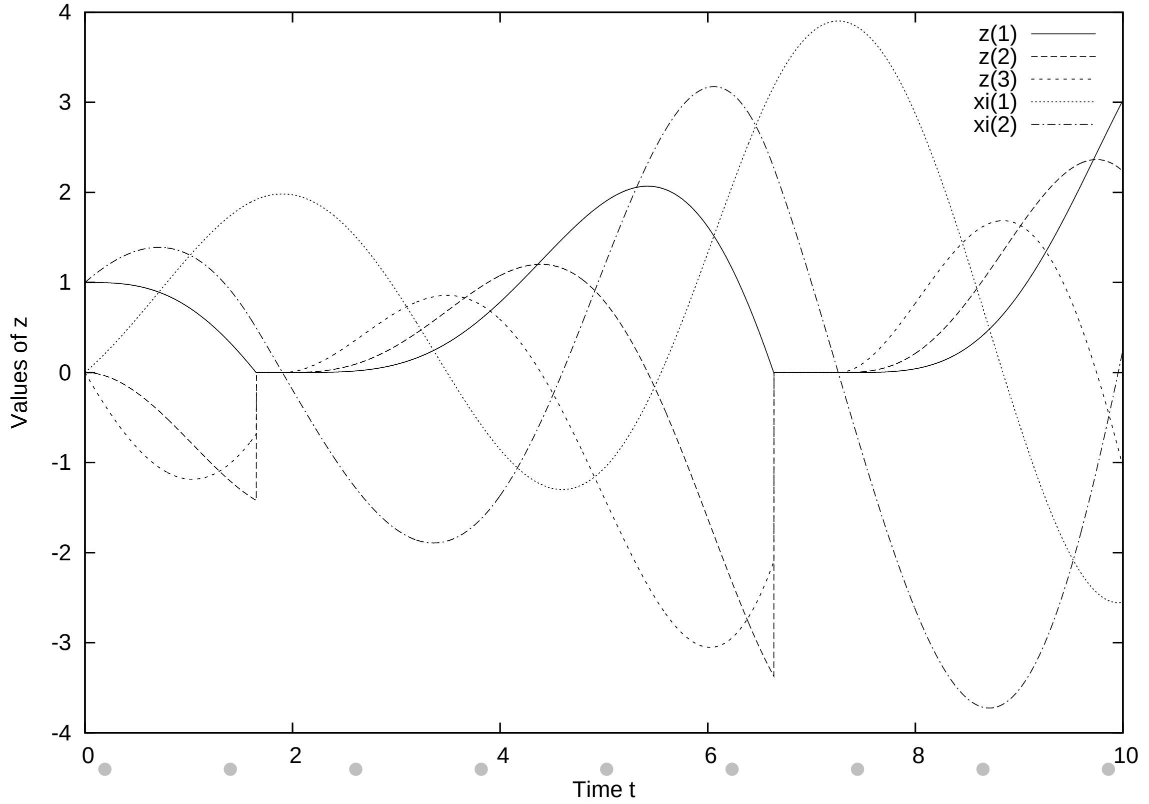


A simple example

A simple example with a non trivial zero-dynamics:

$$\begin{cases} z(0) = (1,0,0,0,0)^T \\ \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = z_3(t) \\ \dot{z}_3(t) = -z_1(t) - z_2(t) - z_3(t) - d_\xi^T \xi(t) + \lambda(t) \\ \dot{\xi}_1(t) = \alpha \xi_2(t) \\ \dot{\xi}_2(t) = -\omega \xi_1(t) + z_1(t) \\ w(t) = z_1(t) \geq 0 \end{cases}$$

For $d_\xi = (0,1)$ the zero dynamic plays role in the global dynamics $\alpha = 1$ and $\omega = 1$.

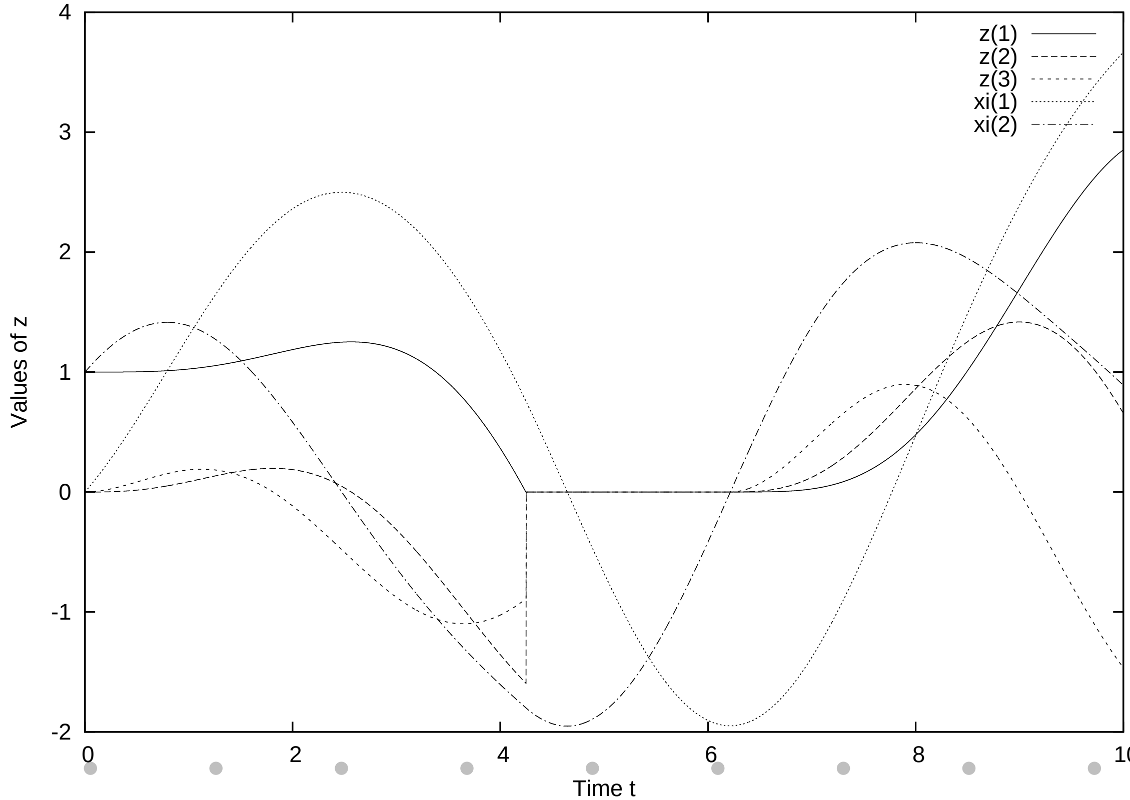


A simple example

A simple example with a non trivial zero-dynamics:

$$\begin{cases} z(0) = (1,0,0,0,0)^T \\ \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = z_3(t) \\ \dot{z}_3(t) = -z_1(t) - z_2(t) - z_3(t) - d_\xi^T \xi(t) + \lambda(t) \\ \dot{\xi}_1(t) = \alpha \xi_2(t) \\ \dot{\xi}_2(t) = -\omega \xi_1(t) + z_1(t) \\ w(t) = z_1(t) \geq 0 \end{cases}$$

For $d_\xi = (0,-1)$ A non trivial active interval is observed $\alpha = 1$ and $\omega = 1$.

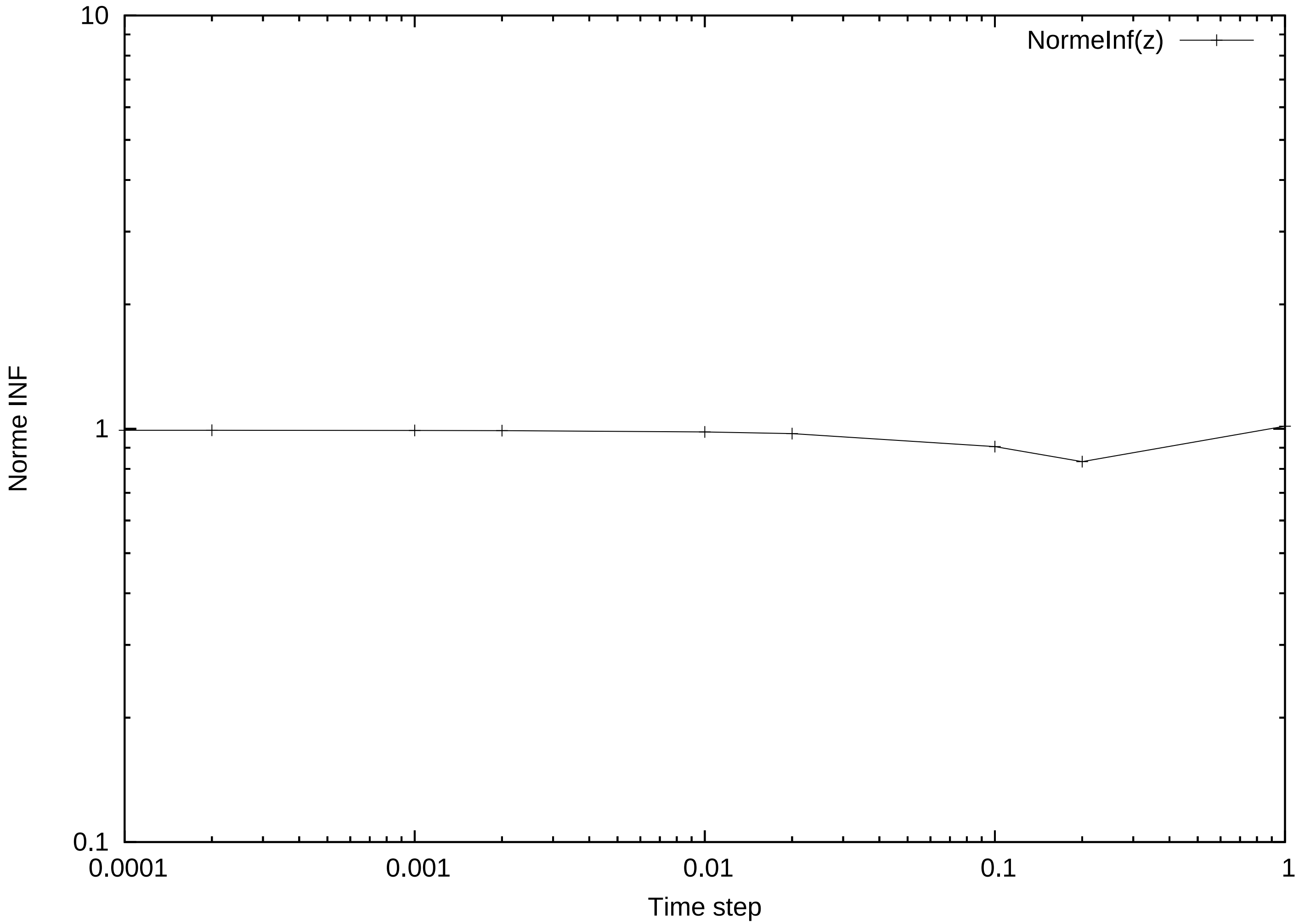


Applications

- Electrical and Mechanical systems with feedback Control Loop
  - The Feedback loop may increase the relative degree of the system
- Indirect Methods for Optimal control with state constraints : Finite difference BVP solvers
  - We can prove that the relative degree of the Necessary condition system is twice the original one of the system to be controlled.
    For a mechanical system (r=2), the necessary conditions for Optimality leads to a dynamical system of relative equal to $r = 4$.
- Advantages of the approach :
  - Take into account accumulation of events.
  - Do not need any first guess for the algorithm
  - Theoretical results

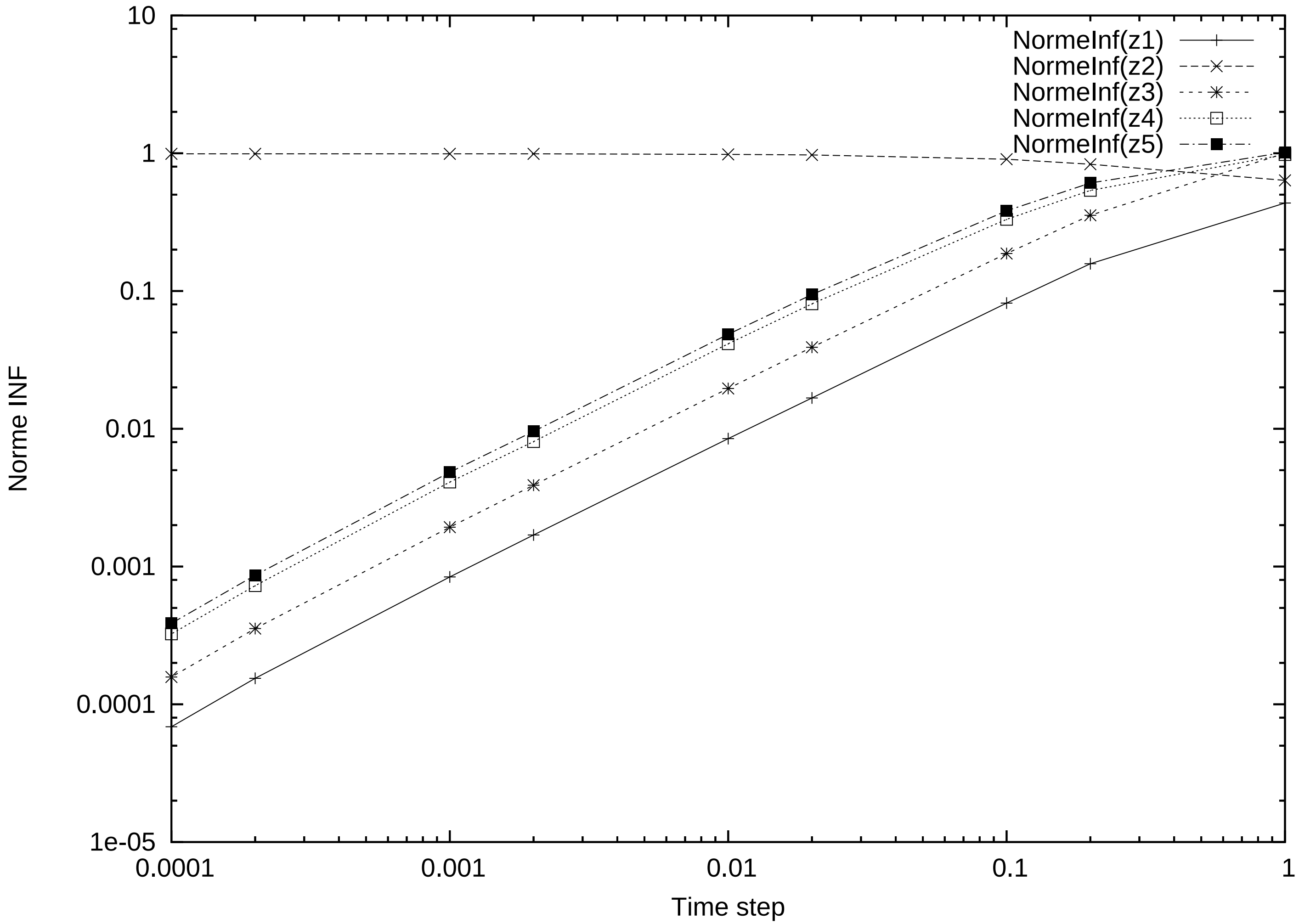
Empirical Order

- This error is measure using the l_∞ norm between the step function generated by the sequences of approximation



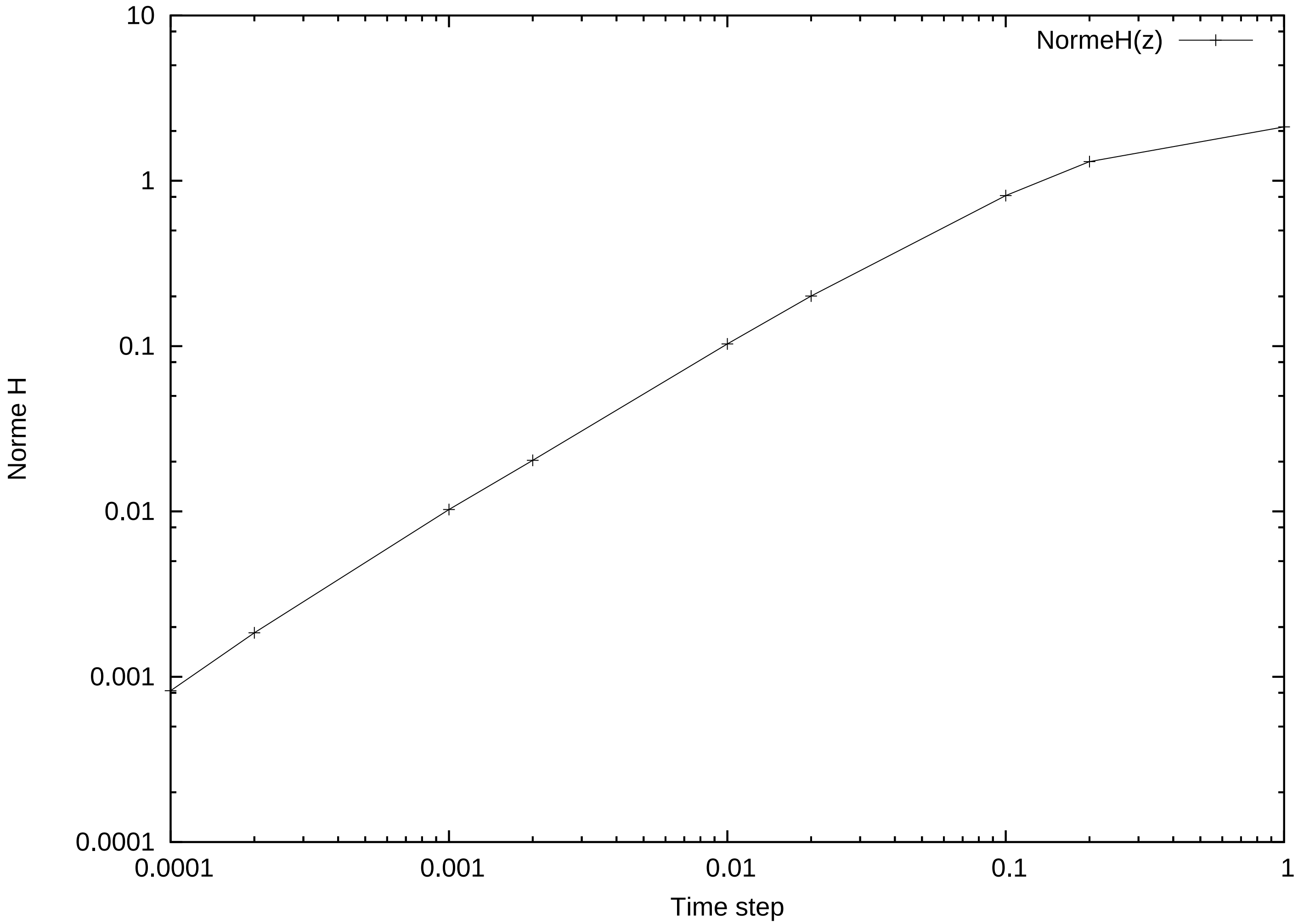
Empirical Order

- This error is measure using the l_∞ norm between the step function generated by the sequences of approximation



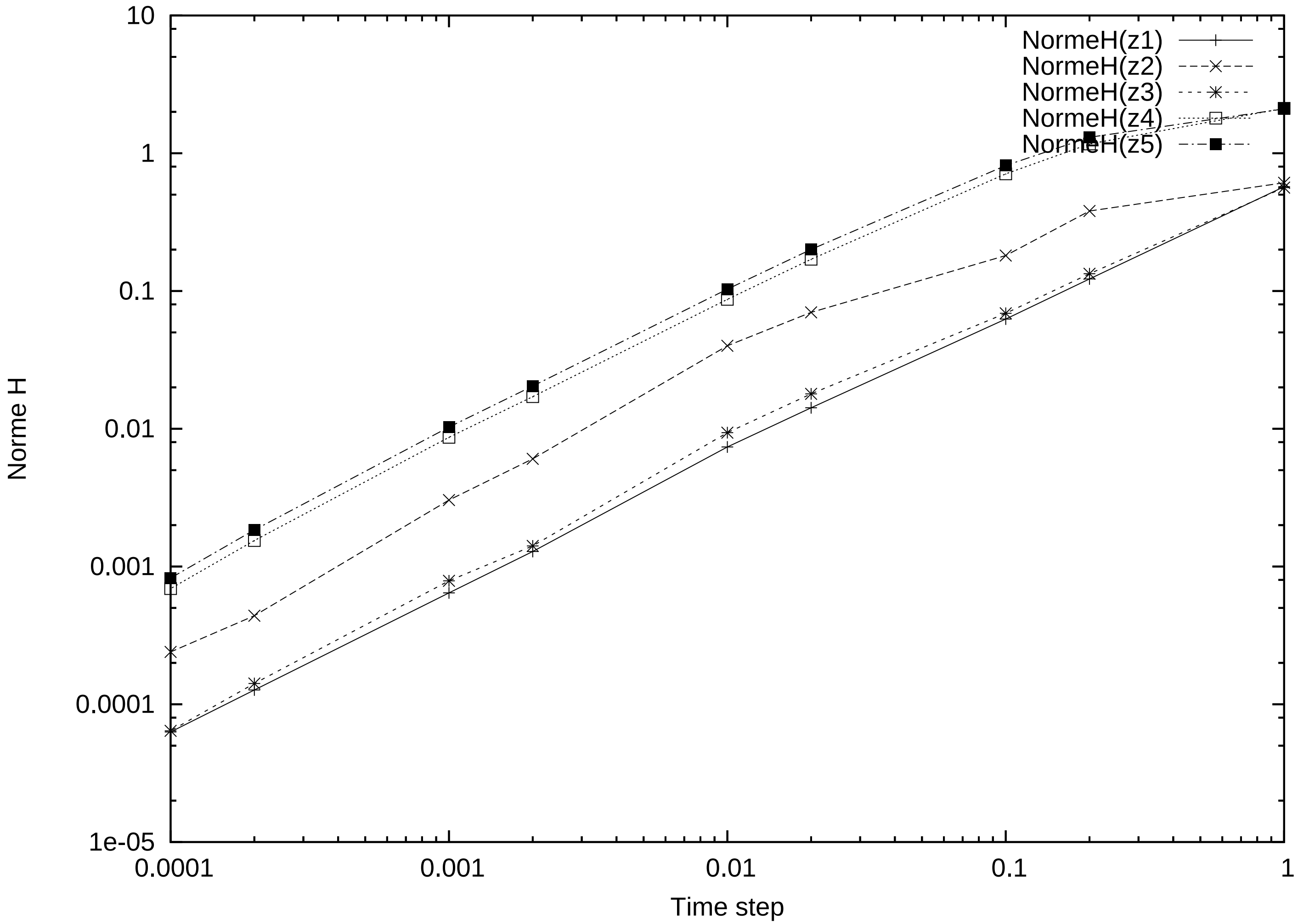
Empirical Order

- This error is now measure using a Hausdorff distance between filled-in graph of BV function.



Empirical Order

- This error is now measure using a Hausdorff distance between filled-in graph of BV function.



Open Problems

- Efficient Algorithms for the Multi-level nested Complementarity problem

$$d\nu_i \in -\partial\psi_{T_{\Phi}^{i-1}(\{Z_{i-1}\}(t^-))}(\{z_i\}(t^+)) \quad \text{on } \tilde{I}, \quad (1 \leq i \leq r)$$

$$\Downarrow$$

$$0 \leq d\nu_1 \perp \{z_1\}(t^+) \geq 0$$
$$\text{if } z_1 \leq 0 \text{ then } 0 \leq d\nu_2 \perp \{z_2\}(t^+) \geq 0$$
$$\text{if } z_1 \leq 0 \text{ and } z_2 \leq 0 \text{ then } 0 \leq d\nu_3 \perp \{z_3\}(t^+) \geq 0$$
$$\text{if } z_1 \leq 0 \text{ and } z_2 \leq 0 \text{ and } z_3 \leq 0 \text{ then } 0 \leq d\nu_4 \perp \{z_4\}(t^+) \geq 0$$
$$\vdots$$

- Non linear and Non autonomous systems
- Higher order Time integration scheme

Outline

Conclusion

There is a lot of stuff to do in the field of
Non Smooth Dynamical systems
....
I would be very grateful if
someone could provide some advises and references
which come from the optimization community

`vincent.acary@inrialpes.fr`