Non Smooth Contact Dynamics approach of cohesive materials

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The main features of the Non Smooth Contact Dynamics method (NSCD) are shortly presented in this paper, the dynamical equation, the Signorini relation as a non smooth modelling of unilateral contact and the frictional Coulomb's law, treated with fully implicit algorithms. By mere changes of variables, it appears that a large class of interface problems, including interface cohesive laws, may be solved using Signorini, Coulomb, standard NSCD algorithms. Emphasis is put on contact between deformable bodies. Illustrating numerical simulation examples are given for fiber reinforced materials, and buildings made of blocks.

Keywords: Contact Dynamics, unilaterality, dry friction, cohesion

1. Introduction

Non smooth mechanics embraces mechanical problems described by steep functions or functions with steep derivatives or non differentiable functions. For instance, an heterogeneous elastic material, composed of stiff elastic inclusions imbedded into a soft elastic medium, is described by a local rigidity 4^{th} order tensor function which appears to be strongly discontinuous from a stiff domain to a soft domain. From the finite element numerical point of view, solving the continuous medium equilibrium equation, leads to solve a linear problem (as far as small perturbations are concerned), involving the elasticity matrix, which happens to be ill conditionned. If the inclusions are nearly rigid bodies, or real rigid bodies, some special technics are to be used, which belongs to the field of non smooth mechanics. A way to deal with a rigid inclusion is to adopt, as in rigid bodies mechanics, a description with 6 degrees of freedom (only 3 in 2 dimensional problems) accounting for translational and rotational motions, and to impose to the boundary nodes of the meshed domain filled by soft material, the rigid velocity field of the inclusion. Though there might be some other ways to deal with rigid inclusions, these considerations enhance the part played by boundary conditions, and might allow to consider more sophisticated boundary conditions as, unilaterality, friction and cohesion. Actually, in this paper, emphasis is put on these kind of boundary conditions.

Another feature of non smooth mechanics, is the inclination to adopt behaviour laws, accounting roughly for the real behaviour, discarding the details, either because insufficient knowledge or data do not allow a meaningful description, or because it is decided that, taking into account those details, will not bring significant changes, and consequently, details should be ignored for simplicity sake. For instance, impenetrability is expressed writing that the gap, the distance g between some candidate point to the antagonist body boundary, is positive, $g \geq 0$. Together with the fact that the reaction force should have a positive normal component $R_N \geq 0$, vanishing if the gap is strictly positive, $gR_N = 0$, one comes to the so called Signorini condition. Another way, coming into details, is to consider that bodies are rough, and that contact is set between elastic asperities scattered on some smooth reference boundaries surfaces. The gap g should be understood as the distance between these reference surfaces, and might take negative values when asperities are deformed. The normal component of the reaction force might appear to depend linearly on the penetration distance, $R_N = -kg$, where k is some stiffness related to the elastic properties of the asperities. The normal reaction force vanishes, if the gap is positive. The stiffnes k should be large enough to forbid too large penetrations, so as to satisfy some 'macroscopic' impenetrability.

In many circumstances, the adopted unilaterality description does not matter, provided that the penetration remains reasonably small. Most often, it is a matter of numerical convenience to adopt one or the other law, the flexibility model, $R_N = -kg$, etc., being a regularized form of the Signorini condition, allowing to use those special methods of non smooth mechanics, which are adapted methods of smooth mechanics. The above discussion, about unilaterality, holds for dry friction. Coulomb's law accounts for the main features of dry friction, i.e. sliding requires that tangential efforts exceed some threshold, the larger is the normal pressure, the larger is the threshold. Regularized forms of Coulomb's law may be relevant as well, and more sophisticated laws might be used, specially if tribological data are available.

Cohesion phenomenon may be defined as the faculty of two bodies to remain into contact as long as separating forces do not exceed some threshold. There exists many physical mechanisms producing such an effect. At molecular scales, Lennard-Jones law is commonly used, accounting for impenetrability, and describing tensile forces rapidly vanishing when the gap is increased, see figure a 1. The relation between the normal reaction force and the gap is smooth but steep. In the line of the

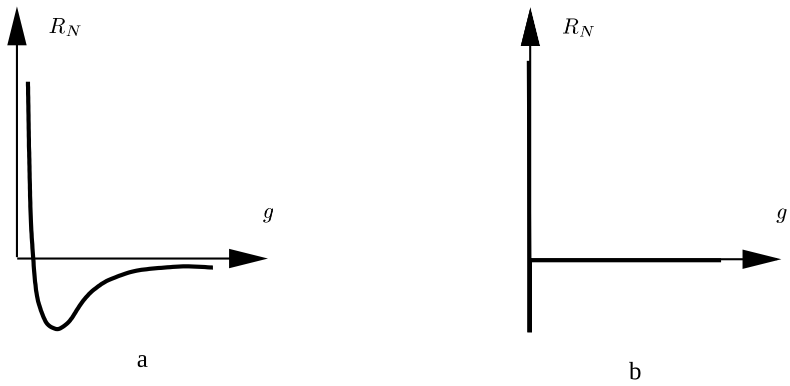

Figure 1. Lennard-Jones graph

above discussion, one may ignore those details describing the cohesive forces near the threshold at a fine scale. One might just keep into consideration, the fact that the gap is positive, that the normal reaction force may not exceed some negative threshold, and that this force vanishes when the gap is strictly positive. This may

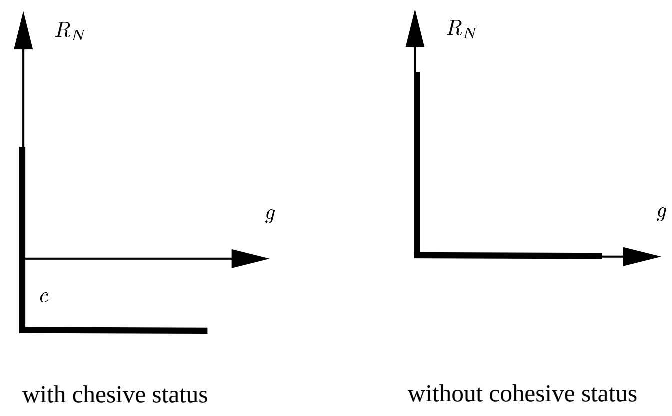

Figure 2. 'Non smooth' cohesive law

be summarized on the graph b figure 1, which may be viewed as an image of the graph b figure 1, resulting from a contraction of the gap scale. In fact, the path on the b graph depends on the loading path, and this b graph does not contains full data. One is lead to distinguish the history of the contacting bodies or *left status* before the contacting instant, and the *right status* (from the numerical point of view, at the beginning and at the end of the time step),

if the left status is 'cohesive':

$g \geq 0\,, \quad R_N + c \geq 0\,, \quad g\,.\,(R_N + c) = 0\,,$

else :

$g \geq 0\,, \quad R_N \geq 0\,, \quad g\,.\,R_N = 0\,,$

the graphs of these relations being depicted figure 2. These relations are to be written together with status change rules:

the right status is the same as the left status,
except,
a) if the left status is 'cohesive',
and if the solution R_N satisfies $R_N + c < 0$ or $g > 0$,
then the right status is set to 'non cohesive',
b) if the left status is 'non cohesive',
and if the solution satisfies $g \leq 0$,
then the right status is set to 'cohesive'.

This rule allows the contacting bodies to be glued again. Other status definitions and change rules may be introduced. For instance, the following rule allows bodies to be glued again only when some pressure is exerted,

b) if the left status is 'non cohesive',
and if the solution satisfies $g \leq 0$,
then the right status is set to 'cohesive'.

The rule b) may be omitted which means that when separated, bodies cannot be glued again. More sophisticated cohesive models will be referred to in next sections, see §2.

In the above section, steep regular models have been presented against strictly non smooth (non differentiable) models, for instance the flexibility model of unilateral constraints against the Signorini relation, the Lennard-Jones model against the Signorini like condition monitored by the cohesive status. Though regularized models allow to use the facilities of smooth methods, adapted for the circumstances, it happens that mere changes of variables allow to apply a 'standard NSCD method' to a wide range of models, so called 'Signorini, Coulomb, derived' models, either steep regular models or strictly non smooth models, as it will be developped in §5 6.

The Non Smooth Contact Dynamics method [JEA 95], [MJ, JJM 92], [JEA 99], [MOR 94], [MOR 99], does not use the facilities of regularized formulations, and rather in the line of Convex Analysis, uses a pair of relations, a complementary relation, the Signorini relation, and Coulomb's law. With this respect, the NSCD method has some relationship with the Newton's generalized method, [ALA 88], [ALA 91], mathematical programming methods, [KLAR 90], [RAO 97], [RAO 99], the bi-potential method, [SAX 91]. Nevertheless, the NSCD solving procedure differs from those used in these methods. The main features of the standard NSCD method, are:

for each candidate to contact (a unique pair composed of a candidate body and of an antagonist body), the relative velocity and the reaction force are related through unilateral Signorini like relation and frictional Coulomb like relation;

the relative velocity and the reaction force are also linearily related through a linearized form of the dynamical equation;

for each candidate to contact, assuming provisional values of the reaction forces at other candidates, a straightforward solution may be computed for the so called 'Signorini, Coulomb, standard' problem defined in the above items;

the unknowns are updated from a candidate to the next candidate using the 'standard' solution; this updating procedure is similar to a non linear block Gauss-Seidel algorithm.

More details will be found in §5 a. It is noteworthy that the NSCD method is fully implicit, while usual regularized methods are fully explicit.

2. Different approaches of cohesive laws

Fracture mechanics and cohesive zone models are two main approaches to analyse crack initiation and propagation in materials.

Rupture mechanics is a classical tool of investigation, particularly successful when applied to homogeneous elastic materials in quasistatic situations. It might fail in more complex circumstances, mainly when comes out the question of initiation of cracks, space (branching) instability and time instability while growing. It might suffer also from the lack of ability to take into account complex physical phenomena such as, frictional contact between the edges of a crack, wear, growth, coalescence, etc. Many efforts have been made in these directions of research. But so far, either theoretically or numerically, describing a complete process of birth and growth of cracks, is still an open problem. Nevertheless, fracture mechanics offers relevant energetic considerations, in particular, the energy release rate G.

The cohesive zone models allow to attack the question of birth and growth of cracks, in complex situations, composite materials, highly heterogeneous microstructured materials, complex local loadings, 3 dimensional cracks, instabilities, etc., while keeping in mind physics at the local scale. Cohesive zone models allow to mix several kinds of energy criterium, such as the critical energy release rate, and the maximal energy criterium (mode II maximal shearing). Crack growth is described with those kinds of local energy criteria, written with normal stresses and opening (gaps) at the edges of the crack. Initiation and propagation should result from the knowledge of the loading path, without supplementary assumptions.

Cohesive zone models, initially introduced by [DUG 60], [BAR 62] and [RIC 68], involved only the relations between normal stresses (traction or compression) and the opening (gap) at the edges of a crack. Later on, shearing was taken into account in the models of [IDA 72] about dynamical fracture in earth crust and [PAL 73] about the shearing of an inclined clay layer under gravity. Unilateral conditions have been introduced later on by [FRÉ 82]. The concept of cohesive domains has been generalized, late eighties, by [NEE 87], who proposed phenomenogical models of decohesion, independant of the loading path, suggested by the atomic scale researches of [ROS 81] and [ROS 83], about bimetallic materials. These models, ready to become realistic crack models, take into account, as soon as 1990, the local irreversible behaviour, introducing surfacic damage, [TVE 90], [NEE 92]. Considering the kinematics of the opening, in the normal direction *as well as* in the tangential direction, [TVE 90] introduces a post-decohesion Coulomb frictional behaviour, and the bounding effect of residual stresses, [TVE 91]. After these pioneering works, recent analytical and numerical developments deal with quasistatic or dynamic propagation of cracks, [XU 94], [CO 9], [XU 98], [NEE 99]. A pair composed of an antagonist point and of a candidate point to contact, in a discrete formulation, might be as well considered as a cohesive zone model. Not only the cohesive zone model applies to cracks but also to inner material. This idea has been applied in quite general situations, elasto-plasticity, [NEE 90], [TVE 92], dynamical propagation of cracks, [XU 94], elasto-visco-plasticity, [SIE 97a], composite materials, [SIE 97], [XU 97],[NEE 99]. One may also quote the contributions of the French school about the micromechanical approach of cracks in composite materials, [FRÉ 82], [MIC 94], [CAN 96], [CHABO 97], and about damage in composite material plies, [ALL 89] , [ALL 95].

3. The basic equations

Though emphasis is put on deformable bodies, the question of space discretization is not discussed in this paper. Finite elements methods are used. Discrete variables and equations governing the nodes coordinates will be written at once.

(*a*) *Notations*

For simplicity sake, it will be assumed, that some nodes on a meshed body candidate to contact are marked as candidate material points. To each candidate P is associated a unit vector $\vec{n}$ orthogonal to the boundary of the antagonist body, directed from the antagonist body to the candidate body. The material point P' lying on the boundary of the antagonist body, and such that $\overrightarrow{P'P}$ is colinear to $\vec{n}$ (i.e. the orthogonal projection of P on the antagonist boundary) is marked as the antagonist material point associated to P. An approximate boundary, or an approximate normal vector is often used, for convenience, and to ensure the uniqueness of the antagonist material point. In tricky circumstances, any reasonable algorithm up to define an antagonist material point P', 'proximal' to P, will do. It will be referred to *a candidate to contact*, as the unique pair of candidate antagonist bodies, or candidate antagonist material point, or merely the candidate material point. An orthonormal basis $(\vec{t}, \vec{n}, \vec{s})$, referred to as the local frame, may be constructed with any two unit vectors $\vec{t}, \vec{s}$, orthogonal to each other and to $\vec{n}$. Normal components of vectors in the local frame, will be noted with the subscript $_N$, while the pair of tangential components will be noted with the subscript $_T$. The relative velocity vector (in an extended sense) is defined as $\vec{U} = \vec{V}(P) - \vec{V}(P')$, where $\vec{V}(P)$ is the velocity vector of the material point P, and $\vec{V}(P')$ is the velocity vector of the material point P'. The components of this vector in the local frame will be written $U = (U_T, U_N)$. The reaction force (dual variable of $\vec{U}$), acting from the antagonist body to the candidate body, has components $R = (R_T, R_N)$, in the local frame. The gap is defined as $g = \overline{PP'}$ and strictly positive when bodies are not contacting, and strictly negative when they interpenetrate. The candidates to contact will be labelled with Greek upperscripts such as $^{\alpha,\beta}$. The number of candidates to contact is χ.

(*b*) *Kinematic relations*

The variable $q \in \mathbb{R}^N$ is the (column) vector of nodes coordinates, n being the number of nodes, $N = 3n$ being the number of degrees of freedom. If some degrees of fredom are imposed, penalisation techniques are used. During a motion, q is a function of the time t. The time derivative of the mapping $t \to q(t)$, is noted $\dot{q}$. There exists a linear mapping $G^\alpha(q)$ such that the relative velocity at the candidate α is given by the formula,

$$U^\alpha = G^\alpha(q)\ \dot{q}\ .$$

For instance, if P is a node candidate to contact, P' the antagnist particle lying on a mesh boundary line with end points, the nodes A, B, The velocity of P' is $\vec{V}(P') = (\overline{AP'}/\overline{AB})\ \vec{V}(B) + (\overline{P'B}/\overline{AB})\ \vec{V}(A)$, and $\vec{U} = \vec{V}(P) - \vec{V}(P')$, which yields a formula as above. From duality considerations (conservation of the power), it comes that the node resulting force r^α corresponding to a local reaction force R^α is,

$$\forall v\ , \quad G^\alpha(q)\ v\ .\ R^\alpha = v\ .\ G^{*\alpha}(q)\ R^\alpha = v\ .\ r^\alpha\ ,$$

where $G^{*\alpha}(q) = H^{\alpha}(q)$ is the linear transposed mapping of $G^{\alpha}(q)$. The usual notation in this paper is,

$$U^{\alpha} = H^{*\alpha}(q)\, \dot{q} \,, \quad r^{\alpha} = H^{\alpha}(q)\, R^{\alpha} \,. \tag{3.1}$$

Another fundamental kinematic formula is that the derivative of the gap function $t \to g(t)$ is the normal component of the relative velocity,

$$\dot{g} = U_N \,. \tag{3.2}$$

(c) *The dynamical equation*

The dynamical equation of the problem is written,

$$M\ddot{q} = F_{int}(q, \dot{q}) + P(t) + r \,, \tag{3.3}$$

where M is the mass matrix, $F_{int}(q, \dot{q})$ represents the node forces corresponding to internal efforts, $P(t)$ represents explicitely known excitation forces, and r are the node forces corresponding to the reaction forces exerted at candidates to contact.

(d) *The Signorini relation, the friction Coulomb's law*

The unilateral Signorini condition writes,

$$g \geq 0 \,, \quad R_N \geq 0 \,, \quad gR_N = 0 \,; \tag{3.4}$$

Using 'standard' variables, U_N, R_N, the complementary relation,

$$\mathrm{U}_N \geq 0 \,, \quad \mathrm{R}_N \geq 0 \,, \quad \mathrm{U}_N \mathrm{R}_N = 0 \,;$$

or any equivalent form, will be referred to as 'Signorini standard' relation,

$$\mathrm{Signor}\,(\mathrm{U}_N, \mathrm{R}_N) \,. \tag{3.5}$$

An equivalent form is,

$$\mathrm{R}_N = \mathrm{proj}_{\mathbb{R}^+}(\mathrm{R}_N - \rho\, \mathrm{U}_N), \quad \text{where } \rho > 0 \text{ is arbitrary} \,. \tag{3.6}$$

Coulomb's law writes,

$$\|R_T\| \leq \mu R_N \,, \quad \|U_T\| \neq 0 \;\Rightarrow\; R_T = -\mu R_N \frac{U_T}{\|U_T\|} \,. \tag{3.7}$$

Using 'standard variables' U, R, the relation,

$$\|\mathrm{R}_T\| \leq \mu \mathrm{R}_N \,, \quad \|\mathrm{U}_T\| \neq 0 \;\Rightarrow\; \mathrm{R}_T = -\mu \mathrm{R}_N \frac{\mathrm{U}_T}{\|\mathrm{U}_T\|} \,,$$

or any equivalent form, will be referred to as 'μ-Coulomb standard',

$$\mathrm{Coul}_{\mu \mathrm{R}_N}(\mathrm{U}_T, \mathrm{R}_T) \,. \tag{3.8}$$

An equivalent form is,

$$\mathrm{R}_T = \mathrm{proj}_{\mu \mathrm{R}_N \mathbb{B}}(\mathrm{R}_T - \rho\, \mathrm{U}_T), \quad \text{where } \rho > 0 \text{ is arbitrary} \,. \tag{3.9}$$

Another equivalent form is expressed as a principle of minimal dissipation,

$$\mathrm{R}_T \in \mu \mathrm{R}_N \mathbb{B} \,, \quad \forall \mathrm{S} \in \mu \mathrm{R}_N \mathbb{B} \,, \quad \mathrm{U}_T.(\mathrm{S} - \mathrm{R}_T) \geq 0 \,. \tag{3.10}$$

$\mathbb{B}$ is the unit disk with center 0 in $\mathbb{R}^2$.

4. Time discretization

(a) Numerical algorithm for the dynamical equation

When time discretization is performed, an elementary subinterval $]t_i, t_{i+1}]$ of length h is considered. The main idea developed in the process of time discretization is that discrete variables are not necessarily to be defined at some special time belonging to this interval, since anyway the times where significant frictional effects occur are usually unknown, or costly to approximate, or even difficult to separate, when simultaneous contacts are occuring. Discrete frictional contact relations are thus defined 'over' the interval of time $]t_i, t_{i+1}]$. Integrating both sides of the dynamical equation yields:

$$\begin{cases} M(\dot{q}(t_{i+1}) - \dot{q}(t_i)) = \displaystyle\int_{t_i}^{t_{i+1}} F(q, \dot{q})\, ds + \int_{]t_i, t_{i+1}]} P dt + \int_{]t_i, t_{i+1}]} r d\nu, \\ \\ q(t_{i+1}) = q(t_i) + \displaystyle\int_{t_i}^{t_{i+1}} \dot{q} ds. \end{cases} \tag{4.1}$$

The *mean value impulse* denoted $r(i+1)$,

$$r(i+1) = \frac{1}{h} \int_{]t_i, t_{i+1}]} r d\nu \,, \tag{4.2}$$

emerges as a primary unknown. Since, in this paper, emphasis is put on cohesive frictional laws, it will be supposed that materials are elastic within small perturbations assumption. Thus technical details will be avoided about nesting some Newton-Raphson loop, to be introduced if the material behaviour law is non linear, see [JEA 99]. A typical method is the θ-method,

$$\begin{cases} \displaystyle\int_{t_i}^{t_i+1} F(q, \dot{q})\, ds \approx \; h\theta F(q(t_{i+1}), \dot{q}(t_{i+1})) + h(1-\theta) F(q(t_i), \dot{q}(t_i)) \,, \\ q(i+1) = q(i) + h\theta \dot{q}(i+1) + h(1-\theta)\dot{q}(i) \,. \end{cases}$$

Assuming an elastic linear behaviour,

$$F(q, \dot{q}) = -V\dot{q} - Kq \,,$$

where V is the damping matrix and K is the stiffness matrix. Setting $\dot{q}(i)$, $q(i)$, $\dot{q}(i+1)$, $q(i+1)$, approximations of $\dot{q}(t_i)$, $q(t_i)$, $\dot{q}(t_{i+1})$, $q(t_{i+1})$, respectively, the equation (4.1) writes:

$$\begin{aligned} M(\dot{q}(i+1) - \dot{q}(i)) =& h\theta(-V\dot{q}(i+1) - Kq(i+1) + P(t_{i+1})) \\ &+ h(1-\theta)(-V\dot{q}(i) - Kq(i) + P(t_i)) + hr(i+1) \,. \end{aligned}$$

and the θ-algorithm is:

$$\left\{\begin{array}{c}\dot{q}(i+1)-\dot{q}(i)= \\ w(-hV\dot{q}(i)-hK(q(i)+\theta h\dot{q}(i))+hP(i+1)+hr(i+1))\,, \\ \\ q(i+1)=q(i)+h\theta\dot{q}(i+1)+h(1-\theta)\dot{q}(i)\,, \\ \\ \text{where} \\ \\ w=(M+h\theta V+h^2\theta^2K)^{-1}\,, \\ P(i+1)=\theta P(t_{i+1})+(1-\theta)P(t_i)\,. \end{array}\right. \tag{4.3}$$

It is assumed that the matrix $M+h\theta V+h^2\theta^2K$ is one-to-one, which is satisfied if h is small enough, the mass matrix M being positive definite. When non smooth mechanics effects, shocks or changes in frictional contact status, are occuring, generating velocities discontinuities, first order schemes are well enough. Higher order approximations schemes are irrelevant and even troublesome, since they require appropriate regularity properties which are not met. The above formula, a linearized form of the discrete dynamical equation, appears as an affine relation between two primary unknowns, the velocity $\dot{q}(i+1)$ and the meanvalue impulse $r(i+1)$,

$$\dot{q}(i+1)=v_{\text{free}}(i)+w\;hr(i+1)\,, \tag{4.4}$$

Many numerical algorithms may be written under such a form. The NSCD method is based on the use of such an affine relation, whatever is the choice of the approximation process for the dynamical equation. In the following, for simplicity sake, the choice $\theta=1$ is made, i.e. the choice corresponding to the implicit Euler method, avoiding consistency problems, see next paragraph.

(*b*) *Discrete forms of kinematic relations*

The adopted discrete forms of (3.1) are,

$$U^\alpha(i+1)=H^{*\alpha}(\hat{q})\;\dot{q}(i+1)\,,\quad r^\alpha(i+1)=H^\alpha(\hat{q})\;R^\alpha(i+1)\,.$$

The variable $\hat{q}$ denotes the configuration used to define the local frames. In a fully implicit method, the choice $\hat{q}=q(i+1)$, should be made. This complication is unnecessary as far as the lenghts $hU^\alpha(i+1)$ remain sufficiently small with respect to curvature radius of contacting points. Actually the choice $\hat{q}=q(i)$ is easy and fair enough. In the following the variable $\hat{q}$ shall be omitted,

$$U^\alpha(i+1)=H^{*\alpha}\;\dot{q}(i+1)\,,\quad r^\alpha(i+1)=H^\alpha\;R^\alpha(i+1)\,. \tag{4.5}$$

Denoting $\dot{g}^\alpha(i+1)$ an approximation of $\dot{g}^\alpha(t_{i+1})$ the formula (3.2) will be writen,

$$\dot{g}^\alpha(i+1)=U_N^\alpha(i+1)\,. \tag{4.6}$$

This formula suggests the predictive formula,

$$g^\alpha(i+1)=g^\alpha(i)+hU_N^\alpha(i+1)\,. \tag{4.7}$$

When using a θ method, $0.5 \leq \theta < 1$, it is recommanded to express unilateral conditions with predictive gaps at times $t_i+(1-\theta)h, t_{i+1}+(1-\theta)h$, i.e. θ-midpoints, instead of end of the time step, with predictive formula such as,

$$g^\alpha(1\text{-}\theta + i) = g^\alpha(i) + (1-\theta)h\, U_N^\alpha(i)\ ,$$
$$g^\alpha(1\text{-}\theta + i + 1) = g^\alpha(i+1) + (1-\theta)h\, U_N^\alpha(i+1)\ .$$

Indeed, unilateral conditions imply a contradictory rule to be satisfied both by the relative velocity and the gap at the end of the time step (except in the case $\theta = 1$). Oscillatory artefacts might be generated. For more details see a discussion about consistency in [JEA 99], [VOL 98].

(c) Discrete forms of Signorini relation, Coulomb's law

The Signorini relation graph is infinitely steep, and consequently the gap g cannot be written as a single-valued mapping of R_N, nor R_N as a single-valued mapping of g. This leads to adopt fully implicit schemes and to choose both the values of $g^\alpha(i+1)$ and $R_N^\alpha(i+1)$ at the end of the time step, as unknowns in the frictional contact laws. For instance the discrete form of Signorini relation (3.4) is,

$$\mathrm{Signor}\,(g^\alpha(i+1), R_N^\alpha(i+1))\ . \tag{4.8}$$

For similar reasons, the discrete form of Coulomb's law (3.7) is,

$$\mathrm{Coul}_{\mu^\alpha R_N^\alpha(i+1)}(U_T^\alpha(i+1), R_T^\alpha(i+1))\ . \tag{4.9}$$

5. The NSCD approach

(a) The 'Signorini, μ-Coulomb, standard' problem

As an example one considers the 'ordinary' dynamical frictional contact problem, governed by the dynamical equation (4.4), the kinematic relations (4.5), the Signorini, Coulomb's relation, (4.8), (4.9).

Using the kinematic relations allows to write a linearized form of the dynamical equation (4.4), in terms of the local variables, the relative velocity $U^\alpha(i+1)$ and the reaction force $R^\alpha(i+1)$, at the candidate to contact α,

$$U^\alpha(i+1) = U^\alpha_{\text{free}} + \textstyle\sum_\beta W^{\alpha\beta}\ hR^\beta(i+1)\ , \tag{5.1}$$
$$W^{\alpha\beta} = H^{*\alpha}\ w\ H^\beta\ , \tag{5.2}$$
$$U^\alpha_{\text{free}} = H^{*\alpha}\ v_{\text{free}}\ . \tag{5.3}$$

Assuming some provisional values for $R^\beta(i+1)$, the above equation writes,

$$U^\alpha(i+1) = U^\alpha_{\text{locfree}} + W^{\alpha\alpha}\ hR^\alpha(i+1)\ , \tag{5.4}$$

where ,

$$U^\alpha_{\text{locfree}} = U^\alpha_{\text{free}} + \sum_{\beta\neq\alpha} W^{\alpha\beta}\ hR^\beta\ ,$$

This equation has to be written together with a discrete form of the Signorini relation, and a discrete form of Coulomb's law, together with the kinematic relation (4.7). Setting,

$$\bar{U}_{locfreeT} = U^{\alpha}_{locfreeT} , \quad \bar{U}_{locfreeN} = U^{\alpha}_{locfreeN} + \tfrac{1}{h} g(i) , \quad \bar{W} = W^{\alpha\alpha} , \quad \mu = \mu^{\alpha} ,$$

it is found that the unknowns,

$$\bar{U}_T = U^{\alpha}_T(i+1) , \quad \bar{U}_N = \tfrac{1}{h} g^{\alpha}(i+1) ,$$
$$\bar{R} = R^{\alpha}(i+1) ,$$

are solutions of the following *'Signorini, μ-Coulomb, standard' problem*,

$$\begin{aligned} &\textit{Signorini, } \mu\textit{-Coulomb, standard problem} : \\ &\text{find } \mathrm{U}, \mathrm{R}, \text{ such that:} \\ &\mathrm{U} = \mathrm{U}_{locfree} + W\, h\mathrm{R} , \\ &\mathrm{Signor}\,(\mathrm{U}_N, \mathrm{R}_N) , \\ &\mathrm{Coul}_{\mu \mathrm{R}_N}(\mathrm{U}_T, \mathrm{R}_T) . \end{aligned} \tag{5.5}$$

where $\mathrm{U}, \mathrm{R}, \mathrm{U}_{locfree} = \bar{U}_{locfree}$ (a data), belong to IR^3 (IR^2 in the 2 dimensional case), $W = \bar{W}$ (a data), is a 3×3 matrix (2×2 in the 2 dimensional case). The solution of this elementary problem is found discussing the intersection of piecewise affine mappings, since Signorini relation and Coulomb's law appears in such a form, see (3.6), (3.9). It happens that in the 2 dimensional case, the elementary standard solution can be explicitely written, see §9, and [JEA 99]. In the 3 dimensional case, a few iterations of a generalized Newton-Raphson method allow super convergence toward the elementary standard solution, [ALA 88]. The NSCD algorithm is as follows:

(*b*) *Solving the frictional contact problem*

The unknowns $R^{\alpha}(i+1)$, average impulses at step $i, i+1$, are sought as the limit of sequences $R^{\alpha}(p)$, $p = 1, ..., p_{max}$, p_{max} maximum number of iterations:

1. At iteration p, seek an approximate solution at candidate α; provisional values at other candidates are adopted as follows; if $\beta > \alpha$, these are the values computed at itration $p-1$ and if $\beta < \alpha$, these are the values just computed at iteration p,

   $$R^{\beta}(p+1),\ \beta = 1, ..., \alpha - 1,\ R^{\beta}(p),\ \beta = \alpha + 1, ..., \chi ;$$

   the free velocity (i.e. the relative velocity at candidate α equiped with a vanishing reaction), while provisional reactions are exerted on neighbouring candidates, is computed,

   $$U^{\alpha}_{locfree} = U^{\alpha}_{free} + \sum_{\beta<\alpha} W^{\alpha\beta}\ hR^{\beta}(p+1) + \sum_{\beta>\alpha} W^{\alpha\beta}\ hR^{\beta}(p) ,$$

(a) compute,

$$\bar{U}_{locfreeT} = U^{\alpha}_{locfreeT} \ , \quad \bar{U}_{locfreeN} = U^{\alpha}_{locfreeN} + \tfrac{1}{h} g(i \ , \quad \bar{W} = W^{\alpha\alpha} \ ;$$

(b) find a solution $\bar{U}, \bar{R}$, of the 'Signorini, μ-Coulomb, standard' problem (5.5), with the data $U_{locfree} = \bar{U}_{locfree}$, $W = \bar{W}, \mu = \mu^{\alpha}$;

(c) set $R^{\alpha}(p+1) = \bar{R}$;

2. reactions are updated and the next candidate is called;

3. the list of candidates is read (p is increased) until some quality criterium is satisfied or $p = p_{max}$.

This method is similar to a non linear block Gauss-Seidel method.

6. Derived 'Signorini, μ-Coulomb, standard' laws

In this section the concept of 'Signorini, μ-Coulomb, derived' law, is introduced: some variables, images of the local basic variables, gap, relative velocity, reaction force, by some affine mappings, satisfy the standard Signorini relation and Coulomb's law. Such changes of variables are defined at the beginning of the time step by some status, a data storage supposed to summarize the significant history of contact.

(a) *Derived law*

A *'Signorini, μ-Coulomb, derived' law* is defined by,

1. *a status variable*, $\mathrm{Status}^{\alpha}(i)$, entirely defined by the relative velocity, the gap, the mean value of the impulse, $U^{\alpha}(i)$, $g^{\alpha}(i)$, $R^{\alpha}(i)$, and possibly status at previous time steps $j < i$;

2. *auxiliary unknowns*, $\bar{U}, \bar{R}$, affine functions of $U^{\alpha}(i+1)$, $g^{\alpha}(i+1)$, $R^{\alpha}(i+1)$; the coefficients of this functions depend on the status $\mathrm{Status}^{\alpha}(i)$ at step i;

3. according to the status, either $R^{\alpha}(i+1) = 0$, or the auxiliary variables $\bar{U}, \bar{R}$, satisfy 'Signorini, μ-Coulomb, standard' law, (3.5), (3.8).

(b) *Change of variables*

The index α is momentarily omitted. It has been seen that the relative velocity and the reaction force are related through the linearized dynamical equation (5.4),

$$U(i+1) = U_{\mathrm{locfree}} + W \ hR(i+1) \ , \tag{6.1}$$

The gap is given by the predictive formula (4.7),

$$g(i+1) = g(i) + h \ U_N(i+1) \ . \tag{6.2}$$

The auxiliary unknowns $\bar{U}, \bar{R}$, are introduced as affine mappings of $U(i+1)$, $g(i+1)$, $R(i+1)$, written,

$$\begin{pmatrix} \bar{U} \\ h\bar{R} \end{pmatrix} = \begin{pmatrix} I_{UU} & I_{UR} \\ I_{RU} & I_{RR} \end{pmatrix} \begin{pmatrix} U(i+1) \\ hR(i+1) \end{pmatrix} + \begin{pmatrix} \bar{U}_0 \\ h\bar{R}_0 \end{pmatrix} \ . \tag{6.3}$$

It is supposed that the matrix,

$$\begin{pmatrix} I_{UU} & I_{UR} \\ I_{RU} & I_{RR} \end{pmatrix} ,$$

is one-to-one. An equivalent form of the previous relation is then,

$$\begin{pmatrix} U(i+1) \\ hR(i+1) \end{pmatrix} = \begin{pmatrix} J_{UU} & J_{UR} \\ J_{RU} & J_{NN} \end{pmatrix} \begin{pmatrix} \bar{U} \\ h\bar{R} \end{pmatrix} + \begin{pmatrix} U_0 \\ hR_0 \end{pmatrix} , \tag{6.4}$$

It comes,

$$\begin{aligned} &(J_{UU} - W J_{RU})\bar{U} = \\ &\qquad (W J_{RR} - J_{UR})h\bar{R} + W\ hR_0 - U_0 + U_{\mathrm{locfree}} . \end{aligned}$$

It is supposed that the matrix $J_{UU} - W J_{RU}$ is one-to-one. The previous relation becomes,

$$\bar{U} = \bar{U}_{\mathrm{locfree}} + \bar{W}\ h\bar{R} , \tag{6.5}$$

where,

$$\begin{aligned} \bar{U}_{\mathrm{locfree}} &= U_{\mathrm{change}} + C_{\mathrm{change}}\ U_{\mathrm{locfree}} , \\ U_{\mathrm{change}} &= C_{\mathrm{change}}\big(W\ hR_0 - U_0\big) , \\ C_{\mathrm{change}} &= (J_{UU} - W J_{RU})^{-1} . \end{aligned} \tag{6.6}$$

It should be noticed that the data J_{UU}, J_{UR}, J_{RU}, J_{RR}, U_0, hR_0, the matrices $\bar{W}$, C_{change}, the vector U_{change}, constructed with physical constants, depend only on α and on $\mathrm{Status}^{\alpha}(i)$. These data are to be prepared at the beginning of the time step, before undergoing iterations. Equation (6.5) is the corresponding form of the linearized dynamical equation (6.1) through changes of variables (6.3), (6.4).

(c) *Solving the frictional contact problem with a derived law*

The algorithm section b is easily generalized.

Prepare the data J^{α}_{UU}, J^{α}_{UR}, J^{α}_{RU}, J^{α}_{RR}, U^{α}_0, hR^{α}_0, $\bar{W}^{\alpha}$, $C^{\alpha}_{\mathrm{change}}$, $U^{\alpha}_{\mathrm{change}}$, for all α and status $\mathrm{Status}^{\alpha}(i)$. The unknowns $R^{\alpha}(i+1)$, are sought as limits of sequences $R^{\alpha}(p)$, $p = 1, ..., p_{max}$:

1. At iteration p, seek an approximate solution at candidate α; provisional values at other candidates are adopted as follows; if $\beta > \alpha$, these are the values computed at itration $p-1$ and if $\beta < \alpha$, these are the values just computed at iteration p,

$$R^{\beta}(p+1),\ \beta = 1, ..., \alpha - 1,\ R^{\beta}(p),\ \beta = \alpha + 1, ..., \chi \ ;$$

compute the free velocity,

$$U^{\alpha}_{locfree} = U^{\alpha}_{free} + \sum_{\beta<\alpha} W^{\alpha\beta}\ hR^{\beta}(p+1) + \sum_{\beta>\alpha} W^{\alpha\beta}\ hR^{\beta}(p) \ ;$$

(a) according to the status $\mathrm{Status}^{\alpha}(i)$, compute $\bar{U}_{\mathrm{locfree}} = U_{\mathrm{change}} + C_{\mathrm{change}}\, U_{\mathrm{locfree}}$,

(b) find a solution $\bar{U}, \bar{R}$, of the 'Signorini, μ-Coulomb, standard' problem (5.5), with the data $U_{locfree} = \bar{U}^{\alpha}_{locfree}$, $W = \bar{W}^{\alpha}$, $\mu = \mu^{\alpha}$;

(c) make the reverse change of variables (6.4) to obtain $R^{\alpha}(p+1)$ from $\bar{R}$,

$$\begin{pmatrix} U^{\alpha}(p+1) \\ \\ hR^{\alpha}(p+1) \end{pmatrix} = \begin{pmatrix} J^{\alpha}_{UU} & J^{\alpha}_{UR} \\ \\ J^{\alpha}_{RU} & J^{\alpha}_{RR} \end{pmatrix} \begin{pmatrix} \bar{U} \\ \\ h\bar{R} \end{pmatrix} + \begin{pmatrix} U^{\alpha}_{0} \\ \\ hR^{\alpha}_{0} \end{pmatrix} ;$$

2. reactions are updated and the next candidate is called;

3. the list of candidates is read (p is increased) until some quality criterium is satisfied or $p = p_{max}$.

The "ordinary" frictional contact problem, presented as an introductory example in §5, using (4.8), (4.9), may be considered as a special case, where a default status is assumed.

(*d*) *Non frictional cohesive example*

As a second example, the NSCD method is applied to the non frictional cohesive law already presented in §1.

Prepare the data:

- if $\mathrm{Status}^{\alpha}(i)$ is 'non cohesive', the following change of variables is adopted, i.e. the 'ordinary' unilateral, frictional model (4.8), (4.9) is used,

$$\begin{aligned} &\bar{U}_T = U^{\alpha}_T(i+1)\,, \quad \bar{U}_N = \tfrac{1}{h} g^{\alpha}(i+1)\,, \\ &\bar{R} = R^{\alpha}(i+1)\,, \\ &\mu^{\alpha} = 0\,, \quad \mu^{\alpha} \text{ friction coefficient}\,, \end{aligned}$$

  It follows,

$$\bar{U}_{locfreeT} = U^{\alpha}_{locfreeT}\,, \quad \bar{U}_{locfreeN} = U^{\alpha}_{locfreeN} + \tfrac{1}{h} g(i)\,, \quad \bar{W} = W^{\alpha\alpha}\,,$$

- if $\mathrm{Status}^{\alpha}(i)$ is 'cohesive', the following change of variables is adopted,

$$\begin{aligned} &\bar{U}_T = U^{\alpha}_T(i+1)\,, \quad \bar{U}_N = \tfrac{1}{h} g^{\alpha}(i+1)\,, \\ &\bar{R}_T = R^{\alpha}_T(i+1)\,, \quad \bar{R}_N = R^{\alpha}_N(i+1) + c^{\alpha}\,, \\ &c^{\alpha} > 0\,, \quad c^{\alpha} \text{ cohesion constant}\,, \\ &\mu^{\alpha} = 0\,, \quad \mu^{\alpha} \text{ friction coefficient}\,, \end{aligned}$$

  It follows,

$$\bar{U}_{locfreeT} = U^{\alpha}_{locfreeT} - W^{\alpha\alpha}_{TN}\, c\,, \quad \bar{U}_{locfreeN} = U^{\alpha}_{locfreeN} + \tfrac{1}{h} g(i) - W^{\alpha\alpha}_{NN}\, c^{\alpha}\,,$$
$$\bar{W} = W^{\alpha\alpha}\,.$$

Some status change rules have to be chosen. The following rules are adopted,

$\text{Status}^{\alpha}(i+1)$ is the same as $\text{Status}^{\alpha}(i)$,
except,
if $\text{Status}^{\alpha}(i)$ is 'cohesive',
and if the solution $R_N^{\alpha}(i+1)$ satisfies $R_N^{\alpha}(i+1) + c^{\alpha} < 0$, or $g^{\alpha}(i+1) > 0$,
then $\text{Status}^{\alpha}(i+1)$ is set to 'non cohesive',

When candidates to contact have been separated, the candidates are not allowed to be glued again. This is the rule in the next applications, §7, §8. Other rules may be adopted.

The case of *Mohr-Coulomb law* is a mere generalization of the above case. One has only to set μ at the value of the friction coefficient μ^{α} in the non cohesive case, and to set $\text{tg}(\varphi^{\alpha})$ in the cohesive case, where φ^{α} is the internal friction angle. The change of status rule is,

$\text{Status}^{\alpha}(i+1)$ is the same as $\text{Status}^{\alpha}(i)$,
except,
if $\text{Status}^{\alpha}(i)$ is 'cohesive',
and if the solution $R^{\alpha}(i+1)$ is such that
$(R_T^{\alpha}(i+1), R_N^{\alpha}(i+1) + c^{\alpha})$ lies outside the Coulomb's cone with φ^{α} opening
or $g^{\alpha}(i+1) > 0$,
then $\text{Status}^{\alpha}(i+1)$ is set to 'non cohesive',

One sees that many laws may be constructed, playing with affine changes of variables and status change rules. Definitions of status may also be refined. This method applies as well to finite elements models of contacting bodies that to collection of rigid bodies as granular materials. More details may be found into [CAM 00], [RAD 00]. As a matter of fact, if the use of (3.4), (3.7), is relevant when one of the contacting bodies is deformable, shock laws are to be used when both contacting bodies are rigid.

(e) *The Frémond, Cangemi, Raous, cohesive model*

The previous model, the Mohr-Coulomb model, is 'very' non smooth. Nevertheless, it is extensively used in soil mechanics, with some refinements, for instance the Cam clay law or the Cambou, J., Sidoroff, model. The concept of damage is often related to some decohesive process. Refining the time scale, introducing damage, is a way to come into details, and to describe more precisely the evolution of interface stresses when decohesion is occuring, i.e. going from undamaged to completely damaged interfaces. Such models are smoother than the mere Mohr-Coulomb model. Furthermore, it is possible to derive such models from thermodynamical considerations. The energy balance between the different contributions is thus easy to identify. In comparison, global energy estimations are possible with 'Signorini, μ-Coulomb, derived' laws, as it is possible for shock laws, but the way the energy is dissipated is purposedly ignored.

The following 'Frémond, Cangemi, Raous' (FCR) model, based on a thermodynamical approach, see [CAN 96], [RAO 97], [RAO 99], belongs to the class of smoothed models in the above sense, i.e. a damage variable is introduced, together with a damage law governing the evolution of the damage variable. Strong velocities variations are expected during the decohesive process and dynamics should be taken into consideration. The damage variable introducing some regularity, one may be inclined to go further in this line, and to derive a completely smooth but steep model, together with explicit numerical methods. Here a different approach is used. It is possible to admit the FCR model in the class of 'Signorini, μ-Coulomb, derived' laws, allowing to use straightforwardly the implicit NSCD method. The FCR model is hereafter presented in this context. At first an intermediate model is considered. The index α is omitted for simplicity.

Prepare the data:

- if $\mathrm{Status}(i)$ is 'non cohesive', the following change of variables is adopted, i.e. the 'ordinary' unilateral, frictional model (4.8), (4.9) is used,

$$\begin{aligned}&\bar{U}_T = U_T(i+1)\,, \quad \bar{U}_N = \tfrac{1}{h} g(i+1)\,,\\ &\bar{R} = R(i+1)\,,\\ &\mu \text{ is the non cohesive status friction coefficient}\,,\end{aligned}$$

- if $\mathrm{Status}(i)$ is 'cohesive', the following change of variables is adopted,

$$\begin{aligned}&\bar{U}_T = U_T(i+1)\,, \quad \bar{U}_N = \tfrac{1}{h} g(i+1)\,,\\ &\bar{R}_T = R_T(i+1) - k_T\; u_T(i+1)\,, \quad (\diamond)\\ &\bar{R}_N = R_N(i+1) - k_N\; g(i+1)\,,\\ &\mu \text{ is the cohesive status friction coefficient}\,.\end{aligned}$$

The symbols k_T, k_N are stiffness constants (depending on α). The variable u_T is the 'tangential displacement', the definition of which is,

$$u_T(t) = \int_{]t_0,t]} U_T(s)ds\,,$$

where t_0 is the instant where the contact is set. Actually, the definition and the numerical estimation of the tangential displacement rise serious difficulties. Within the small perturbations assumption, with finite sliding between small curvature bodies, the following approximate formula may be adopted,

$$u_T(i+i) = u_T(i) + hU_T(i+1)\,, \tag{6.7}$$

so that the above formula quoted ($\diamond$) may be written,

$$\bar{R}_T = R_T(i+1) + k_T\; u_T(i) + hk_T\; U_T(i+1)\,.$$

If the variables $\bar{U}$, $\bar{R}$, satisfy the 'Signorini, μ-Coulomb, standard' law, the variables, $R_N(i+1), \frac{1}{h}g(i+1)$ lie on the graph figure 3 a), and the variables $R_T(i+1), U_T(i+1)$

lie on the graph figure 3 b). It means that, when the status is cohesive, some spring with stiffness k_N is acting from the actual antagonist point to the actual candidate point in the normal direction to prevent separation, while some spring with stiffness k_T is attached from the initial antagonist point (at t_0) to the actual candidate frictional point in the tangential direction. The FCR model is more sophisticated

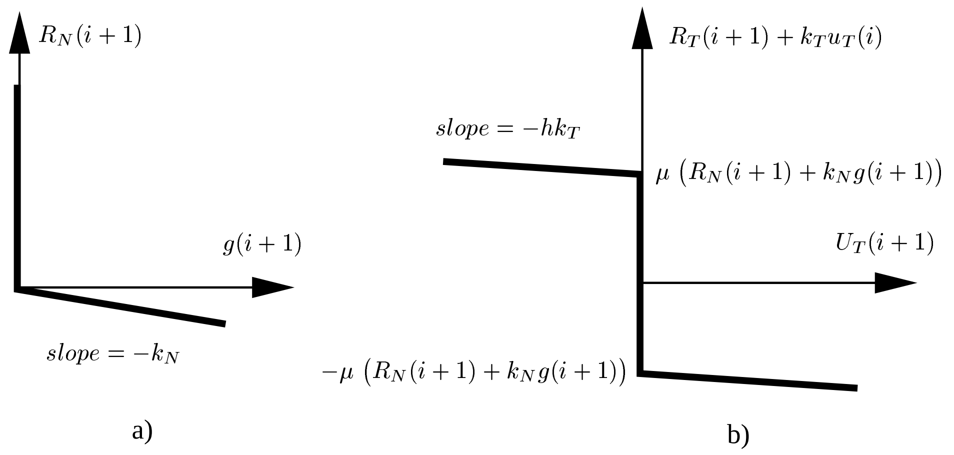

Figure 3. a) unilateral condition, b) friction law

in the sense that the stiffness k_N, k_T, are degradating according to some damage variable β governed by a differential equation, with discrete form,

$$
\begin{aligned}
&k_T = \beta(i+1)^2\ C_T\ , \quad k_N = \beta(i+1)^2\ C_N\ ,\\
&\beta(i+1) = \beta(i) + h\dot{\beta}(i)\ ,\\
&\dot{\beta}(i) = -\big(\frac{1}{b}[EG'(\beta(i)) - \beta(i)(C_N\ g^2(i) + C_T\ u_T^2(i))]^-\big)^{1/p}\ , \quad \text{if } \beta \in [0,1[\ ,\\
&\dot{\beta}(i) \leq -\big(\frac{1}{b}[EG'(\beta(i)) - \beta(i)(C_N\ g^2(i) + C_T\ u_T^2(i))]^-\big)^{1/p}\ , \quad \text{if } \beta = 1\ ,
\end{aligned}
$$

The notation $[x]^-$ stands for the negative part of x, $[x]^- = \max(0, -x)$. The symbols E, b, C_T, C_N are physical constants (depending on α). The variable $\beta \in [0,1]$ is the damage variable. If $\beta = 0$ the interface in the vicinity of the candidate to contact is undamaged, and is fully damaged if $\beta = 1$. For simplicity sake assume $G'(\beta(i)) = 1$. The symbol E is some reference energy. One sees from the above formula that as soon as the springs elastic energy, $\frac{1}{2}(C_N\ g^2(i) + C_T\ u_T^2(i))$ reaches the value E, the damage variable begins to decrease and keeps on decreasing as long as $\beta(i)^2\frac{1}{2}(C_N\ g^2(i) + C_T\ u_T^2(i)) \geq E\beta(i)$. The status change rule is: if $\beta \leq 0$, then $\mathrm{Status}(i)$ is set to 'non cohesive'. In fact the 'cohesive' behaviour law degenerates as the 'non cohesive' law as soon as the damage is achieved, $\beta \leq 0$.

The following example shows the possibilities of the FCR model operated by the NSCD method.

7. An exemple of fiber reinforced material

The figure 4 illustrates a typical behaviour of a (Sic/Sic) composite, reinforced with long fibers. Half a representative cell, including matrix and a 15 μ diameter fiber, is submitted to periodic limit conditions allowing to account for the effects of a crack within the matrix, lying in a plane orthogonal to the fibers bundle. This configuration, together with the FCR model, allows to analyse three main phenomena physically observed during the crack process: the crack propagates around the fibers; some bridging appears between fibers and matrix, the crack propagates more slowly or even stops, being captured by the fibers bundle; the crack breaks through matrix and fiber. These intrinsically 3 dimensional phenomena, depend on the respective interface matrix-fiber, matrix-matrix, fiber-fiber properties, some of them happening to be optimal from the point of view of the ability of the composite material to overcome crack propagation.

In the figure 4, the cell is submitted to given tensile displacements imposed to the upper section in the direction of fibers. Different physical constants are used to model the frictional decohesive process matrix-fiber, and fiber-fiber, matrix-matrix, at a decohesive zone located at the lower section. In this example, the chosen values allow a progressive decohesive process at the fiber-matrix interface, and a britle fracture process at the lower section, across matrix. In this particular case, one observes a crack in the matrix bridged by the fiber: since fiber-matrix cohesive properties are poor, the matrix crack is soon deviated at the fiber-matrix interface, (white: $\beta = 1$, black: $\beta = 0$). The fiber, where becoming unstuck from the matrix, is still extracted with friction. This dissipative process results in a global ductile behaviour of the composite material. More details may be found in [MON 00].

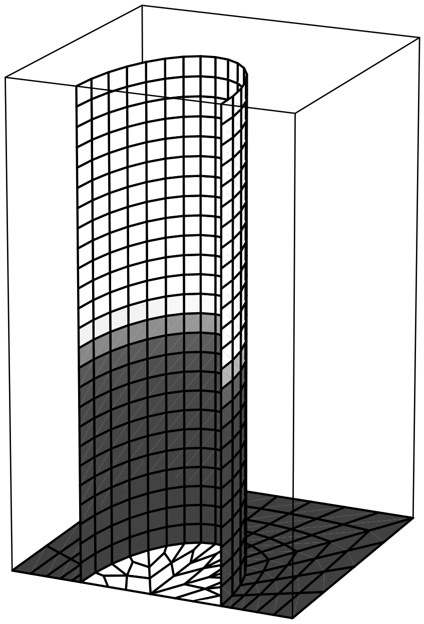

Figure 4. Fiber reinforced material: representative cell, damaged zones

8. An exemple of building made of blocks

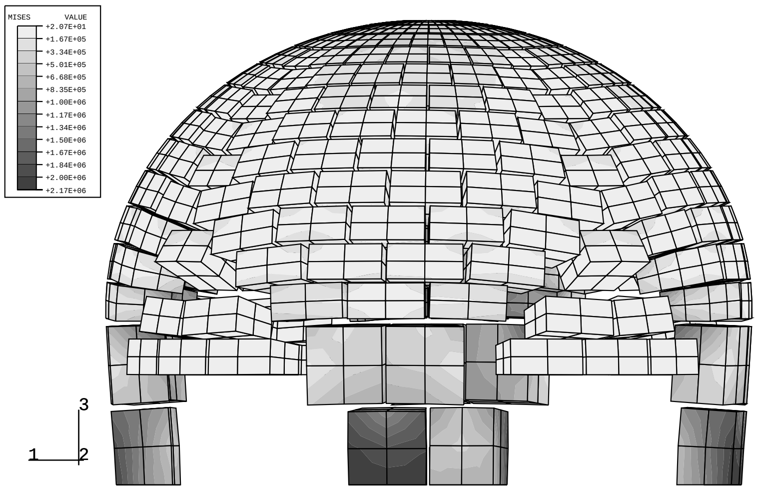

Figure 5. Non cohesive dome

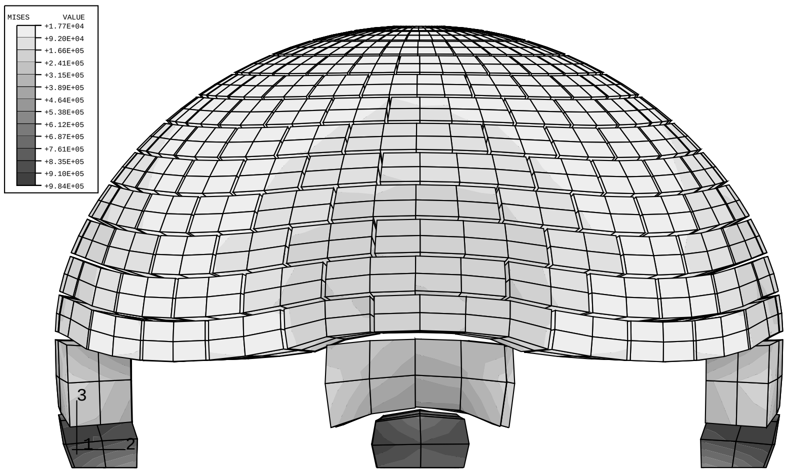

Figure 6. Cohesive dome

In the following example, a numerical test dome, 10 m diameter, made of 300 granit elastic blocks, set on 4 pilars, is submitted to gravity. Each block is composed of 8 H8 finite elements. The structure is composed of 2400 H8 elements, 8100 nodes, 24300 degrees of freedom. Each block face owns $4 \times 4 = 16$ candidate points to contact. The total number of candidates to contact is 9176. The adopted cohesive frictional law is the Mohr-Coulomb law. The figures, 5, and 6 from a slightly different view angle, show the bonding with thick joints (actually numericaly infinitely thin), and the mesh. The gravity is applied to an unstressed initial configuration. An equilibrium is reached when some highly damped transient state has vanished. In the figure 5, blocks are laid without any cohesive properties. One sees that some blocks are beginning to fall between pilars (displacements are magnified 2. 10^4 times). In the second example, some cohesion has been introduced, as much as to allow any joint to resist a tensile force equal to the weight of six blocks, enough to ensure the ability of six blocks layers between pilars to resist gravity. In these circumstances the dome is stable (displacements are magnified 3. 10^5 times). The scale shows the Von Mises stresses and particularly, compression in the pilars (black) and traction in layers between pilars (white). See [ACA 98], [ACA 00].

9. Annex: the 2 dimensional 'Signorini, μ-Coulomb, standard' problem solution

One sets:

$$W = \begin{pmatrix} W_{TT} & W_{TN} \\ W_{NT} & W_{NN} \end{pmatrix} \,,$$

$$hR_{locstick} = -W^{-1}\, U_{locfree} \,;$$

shorter notations are adopted,

$$U_{free} = U_{locfree} \,,\; R_{stick} = R_{locstick} \,.$$

It is supposed that W is positive definite. The term $U_{free}, (U_{locfree})$ is the free local velocity, i.e. the local velocity when reaction is discarded on the concerned contact, $R = 0$; the term $R_{stick}, (R_{locstick})$ may be viewed as the reaction when a null relative velocity is imposed, $U = 0$.

$$\text{It is supposed } -1 < -\mu \frac{W_{NT}}{W_{NN}} < 1 \,. \tag{9.1}$$

the *standard solution is* , (9.2)

if $U_{freeN} > 0$, then: no contact,

$hR = 0$;

if $U_{freeN} \leq 0$ and $R_{stickT} + \mu R_{stickN} < 0$, then: forward frictional sliding,

$$hR_T = -\mu hR_N \ , \ hR_N = -\frac{1}{(1 - \mu \frac{W_{NT}}{W_{NN}}) W_{NN}} \ U_{freeN} \ ;$$

if $U_{freeN} \leq 0$ and $R_{stickT} - \mu R_{stickN} > 0$, then: backward frictional sliding,

$$hR_T = \ \mu hR_N \ , \ hR_N = -\frac{1}{(1 + \mu \frac{W_{NT}}{W_{NN}}) W_{NN}} \ U_{freeN} \ ;$$

if $U_{freeN} \leq 0$ and $R_{stickT} + \mu R_{stickN} \geq 0$ and $R_{stickT} - \mu R_{stickN} \leq 0$,

then: sticking,

$hR = hR_{stick}$.

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