Toward a multiple impact law: the 3-ball chain example.

V. Acary, B. Brogliato

INRIA Rhône-Alpes, ZIRST, 655 avenue de l'Europe, Montbonnot
F-38334 Saint Ismier Cedex
Vincent.Acary@inrialpes.fr, Bernard.Brogliato@inrialpes.fr

Abstract

The aim of this work is to exhibit a multiple impact law for rigid body dynamical system which meets the properties of closing the non-smooth dynamical equations and of corroborating experiments. This law is based on the *impulse correlation ratio* which is computed from equivalent regularized model with compliant contact. A case-study on 3-ball chain and n-ball chain are delineated and results on finite dimensional system are stated.

1 Introduction and motivations

Multiple impact events appear in finite dimensional mechanical systems with unilateral constraints when a set of degree of freedom are subjected to at least two independent constraints at the same time. Chains of balls or the Newton's cradle are academe-mic examples of systems where concurrent multiple impacts occur. In more realistic situations, such as impact in mechanisms or dense granular flows, multiple impacts play an important role.

In a classical single impact case, the definition of an *impact law* allows one to compute the post-impact velocity in a unique way with some mathematical cares on the data [2 ; 7]. For multiple impact events, there are still two open problems: the formulation of a general multiple impact law and the continuous dependence with the respect to initial data. The first problem of the *formulation of a multiple impact law* is addressed in this paper.

A reliable multiple impact law must possess the following properties :

1. It must be a mapping that completely determines, in a unique way, the post-impact velocities of the system knowing the pre-impact velocities and the present configuration.
2. This mapping must respect the fundamental principles of mechanics and thermodynamics. Particularly, unilateral constraints and energy balance have to be satisfied.

3. It corroborates the experimental observations, and the set of parameters which constitute the law, must be measurable and physically justified. Better, the parameters of the law may be correlated with the geometrical and the material characteristics of the bodies involved in the impact.

4. It must be numerically tractable. The post-impact velocities must be computed through an efficient numerical procedure. In order to achieve this point, the multiple impact must be formulated in a non-smooth minimization problem or complementarity problem.

The aim of this work is to exhibit an impact law which meets the preceding conditions, for a particular class of systems.

When multiple impacts occur, most of classical formulations (Newton, Poisson, energetic coefficient) do not respect all of the requirements 1-4. The algorithm of Han and Gilmore [9] provides a good energetic treatment but the existence of a solution is not guaranteed [3]. Moreau [12] and Glocker and Pfeiffer [8] propose two impact laws, numerically efficient, which always provide a solution, but the post-impact velocities are not always satisfying from an experimental point of view. Frémond [6] presents an elegant and rigorous framework to add internal constraints in mechanical systems, which are consistent with thermodynamic principles. However, a precise physical definition of the parameters of such laws somewhat lacks. Motivated by an experimental work on Newton's cradle, Ceanga and Hurmuzlu [4] postulate the existence of an *impulse correlation ratio* (ICR) α for a triplet of balls. With the help of energetic restitution coefficients, the post-impact velocities are experimentally shown to be well approximated. In this last work, it is not clear that the properties 1 and 4 are satisfied.

In fact, the problem of the propagation of the impulse in a chain of body in contact is hidden behind a definition of a multiple impact law. In the general case, it seems bard to treat such phenomena with rigid body model, since this propagation of impulses is essentially due to the vibration of the bodies involved in the impact. However, in particular cases such as hard balls chains, or in general, when the assumptions of the Hertz contact theory are satisfied, there is no bulk vibrations of the bodies and the deformation is located near the contact point. Then, the chain of balls works as a line of discrete mass interconnected with massless nonlinear springs (for more details, see Falcon et al. [5]). In this paper, we shed new light on the ICR by studying the regularized system of a 3-ball chain with elastic contact springs.

The industrial application of this work is led through a fruitful collaboration with M. Abadie [1] from Schneider Electric, concerning the virtual prototyping of circuit breaker mechanisms, where a fine modelling of impact is an essential step.

2 Case study of a 3-ball chain regularized with elastic springs

In this section, we focus our attention on 3-ball chains, which are very interesting examples of systems with multiple impacts. Furthermore, Hertz theory of contact is very well correlated with the experiments at low velocity range [5]. This remark justifies physically the use of a regularized model based on Hertzian springs.

2.1 Rigid body model of a 3-ball chain

A dynamical system of three rigid balls of equal mass m, described by their center of mass positions q_1, q_2, q_3 and velocities v_1, v_2, v_3 is considered. Each ball slides without friction on a straight line and the dynamics at the instant of impact is:

$$\begin{cases} m(v_1^+ - v_1) = -p_1 \\ m(v_2^+ - v_2) = p_1 - p_2 \\ m(v_3^+ - v_3) = p_2 \end{cases} \tag{1}$$

where v_i, v_i^+ are respectively the pre-impact and the post-impact velocities and p_j the impulses. Without loss of generality, the pre-impact velocity of the middle ball is chosen equal to zero ($v_2 = 0$). An additional law is given to address the energetic behaviour at impact:

$$(v_1^+)^2 + (v_2^+)^2 + (v_3^+)^2 \quad = \quad e\,(v_1^2 + v_3^2) \tag{2}$$

where e is a total energetic coefficient. If a multiple impact occurs (i.e. the three balls are in contact at the same instant), this system is not mathematically well-posed. Indeed, for $[v_1, v_3] = [1, 0]$, one can easily check that $[v_1^+, v_2^+, v_3^+] = [0, 0, 1]$ and $[v_1^+, v_2^+, v_3^+] = [-1/3, 2/3, 2/3]$ can be solutions of this system for the conservative case ($e = 1$).

For a given value of the ICR, $\alpha = \dfrac{p_1}{p_2}$, the system becomes well-posed and the post-impact velocities are determined uniquely (see Appendix A of the extended paper on cd for details of calculation).

2.2 Numerical experiments

Let us consider an equivalent regularized system for the 3-ball chain. The interaction between two balls is no longer rigid but realized through an Hertzian spring model. We are interested in relative motion between the balls, therefore we choose to write down the dynamical system in terms of indentations, $\delta_i = q_{i+1} - q_i$, as :

$$\begin{cases} m\ddot{\delta}_1 = -2f_1(\delta_1) + f_2(\delta_2) \\ m\ddot{\delta}_2 = -2f_2(\delta_2) + f_1(\delta_1) \\ 0 \leq \boldsymbol{f} \perp \boldsymbol{f} - \boldsymbol{K}(\boldsymbol{\delta}).\boldsymbol{\delta} \geq 0 \end{cases} \tag{3}$$

where $\boldsymbol{f} = [f_1, f_2]^T$ represents the efforts between balls, $\boldsymbol{\delta} = [\delta_1, \delta_2]^T$ the vector of collected indentations and $\boldsymbol{K}(\boldsymbol{\delta})$ is the stiffness matrix. For Hertzian contact, the stiffness matrix takes the form :

$$K(\boldsymbol{\delta}) = \begin{bmatrix} k_1\delta_1^{1/2} & 0 \\ 0 & k_2\delta_2^{1/2} \end{bmatrix} \tag{4}$$

where $k_1 = k$ and $k_2 = \kappa k$, $\kappa \in \mathbb{R}_+$ are the coefficients of stiffness related to some material and geometrical parameters.

The integration, which is intractable analytically, is performed with Scilab© for various initial relative velocities (choosing $v_2 = 0$). Actually, the solution is sufficiently smooth to allow the use of a traditional numerical ODE solver. On Figure 1, some curves are given which draw the forces between balls versus time. One can remark that the process of collision is not trivial: several periods of contact may occur before the balls separate definitively (*see* Figure 1(b)1(c)), or the contact period between two balls may not begin at the first instant of contact (*see* Figure 1(d)).

If we define a multiple impact in regularized systems as the existence of a time interval where both contact forces are different from zero, all of these processes lead to multiple impacts. Naturally, the rigid limit in a mathematical sense requires additional care.

2.3 Preliminary results.

Numerical simulations have also be carried out with linear springs (*see* Figure 1(e)-1(h)). This model allows one to compute analytical response of the systems based on a modal analysis (see Appendix B of the extended paper on cd for details of calculation). It is noteworthy that the occurrence of transcendental equations in the resolution creates serious difficulties to integrate analytically the process of collisions. Particularly, the time and the order of interactions are not easily predictable.

Nevertheless, a preliminary conclusion can be stated for linear springs :

Proposition 2.1 The instants of changes in the contact interactions, in an adimensional scale of time, for instance, $T = \omega_i t$ (where ω_i is a modal frequency of the system), and the ratio of impulses, α, do not depend on the absolute values of stiffness k and mass m. Moreover, the impulse correlation ratio α, is completely determined by the natural modes of the regularized dynamical system and the pre-impact velocities.

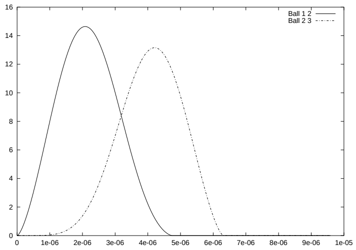
This conclusion outlines two important consequences:

- from a mechanical point of view, the introduction of an impulse ratio enhances the model with some informations about the behavior of dynamical system when it is binded by elastic contact,
- from a numerical modelling point of view, the independence to absolute value ok k allows one to consider in a consistent manner its applications to very large stiffnesses, which are generally encountered in applications.

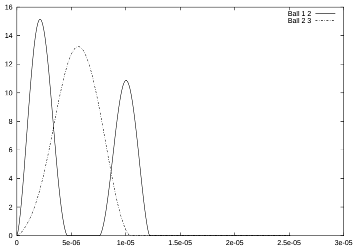
3 Some remarks on impulse correlation ratios in n-ball chains

An important aspect of a correct impact law is that it qualitatively represents the physical phenomena. For the n-ball chain or the Newton's cradle, we know that conservation of kinetic energy and momentum is not sufficient to explain that there is no ball at rest after an impact [10]. The introduction of a set of ICR in n-ball chain as Ceanga and Hurmuzlu [4] have done, describes qualitatively this important phenomenon.

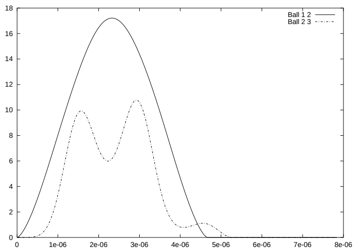
From a quantitative point of view, some remarks must be made. Let us study the values of the ICR obtained by numerical simulation of a n-ball chain made of steel ($E = 210\mathsf{Mpa}, \nu =$



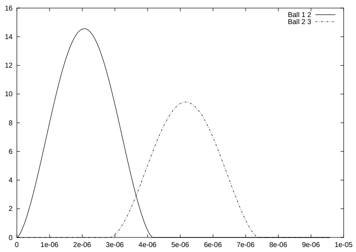
(a) $\kappa = 1, v_1 = 1, v_3 = 0$



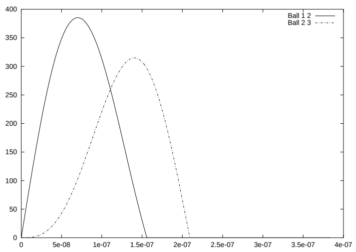
(b) $\kappa = 0.1, v_1 = 1, v_3 = -1$



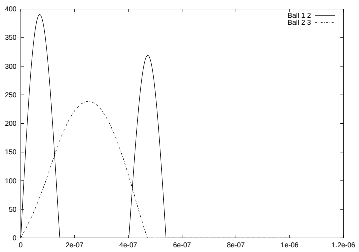
(c) $\kappa = 100, v_1 = 1, v_3 = 0$



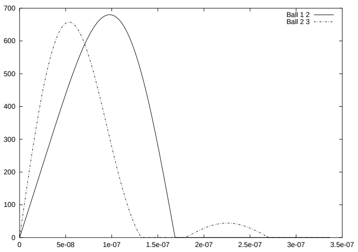
(d) $\kappa = 1, v_1 = 1, v_3 = 0.3$



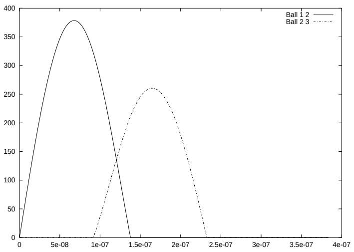
(e) $\kappa = 1, v_1 = 1, v_3 = 0$



(f) $\kappa = 0.1, v_1 = 1, v_3 = -1$



(g) $\kappa = 2.24, v_1 = 1, v_3 = -1$



(h) $\kappa = 1, v_1 = 1, v_3 = 0.3$

Figure 1: Numerical integration of 3 balls chain. Forces between balls versus time. Figures (a-d) Hertzian spring contact. Figures (e-h) linear spring

.

$0.3, \rho = 7800\mathsf{kg}/\mathsf{m}^3$) regularized with elastic Hertz model, where the first ball is dropped at $1m/s$ and the other balls are at rest.

On the Figure 2(a), the number of balls of radius $10\mathsf{mm}$ in the chain ranges from 3 to 21. For n balls, there are $n-1$ impulses and $n-2$ ICR, defined by:

$$\alpha_i = \text{icr}(i) = \frac{p_i}{p_{i+1}} \tag{5}$$

The first remark is that the ICR which corresponds to the last triplet in the chain (for instance, the point A for 8 balls) is very different from the others. Therefore, the value of ICR measured from an experiment on a triplet cannot be used for the n-ball chain.

On the Figure 2(b), we observe the value of ICR in an 21-ball where the tenth ball has been changed to a big ball of radius $50\mathsf{mm}$. The ICR corresponding to the percussion on the big ball is different, but also the value of ICR for the triplets 10 to 19. Moreover, the value of ICR computed for a 3-ball with a middle big wall is about 63.47, which is very different from the value computed in the whole chain (point B). This shows that the ICR depends on the dynamical features of the whole coupled system.

4 Toward an extension to finite dimensional systems – Major results and conclusion

The case study of 3-ball chain is extended to finite dimensional systems subjected to perfect unilateral constraints. The major results are :

1. The post-impact velocity, computed with the multiple impact law defined by *impulse correlation ratio*, augmented by a total energetic law (2), is provided in a unique way and the system becomes mathematically well-posed.

2. If the perfect constraints are regularized by a general viscoelastic contact model corresponding to a linear viscoelastic bulk behavior [11 ; 13] i.e. $f = K\delta^n + C\delta^{n-1}\dot{\delta}$ then

   (a) The ratio of impulse is finite and the subspace of the state space defined by

   $$E = \{\delta \geq 0, \dot{\delta} \geq 0\} \tag{6}$$

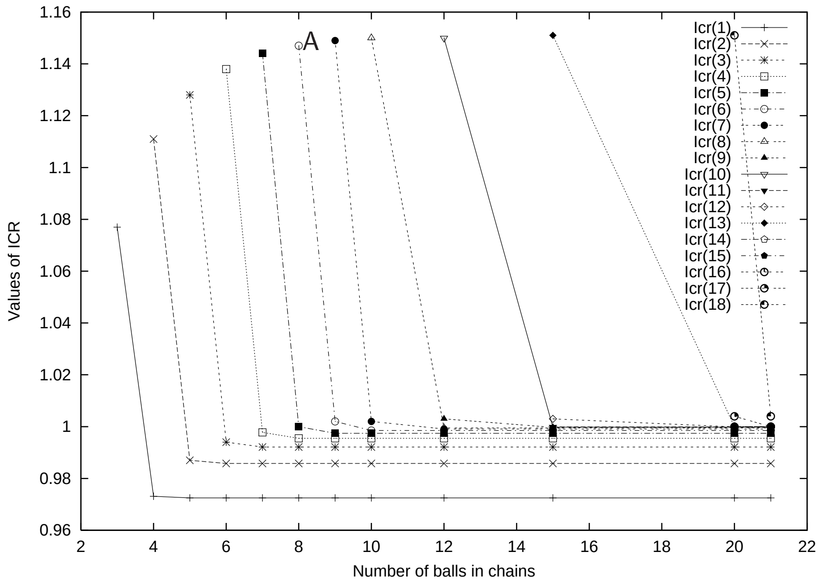
   is globally attractive. Moreover, the amplitude of the force asymptotically tends towards zero and the relative velocity $\dot{\delta}$ towards a finite constant. This last point is very important from a numerical point of view. Extending these results to finite time convergence is still an issue.

   (b) The ICRs are independent of the absolute value of stiffness.

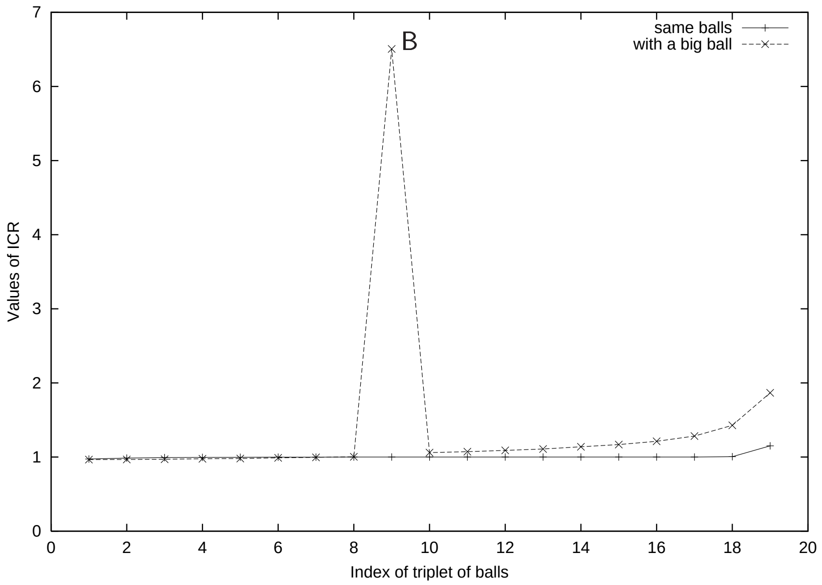
3. If the perfect constraints are regularized by a linear model, i.e.

   $$f = K\delta \tag{7}$$

   then the ICRs depend only on natural modes of the system and the pre-impact velocities.



(a) ICR versus the number of balls in the chain



(b) ICR versus the index of triplet in the 21-ball chain - Comparison between chain of same balls and a chain with a big ball 10

Figure 2: Impulse correlation ratios in a n-ball chain

By way of conclusion, we can say that the introduction of a set of ICR allows one to compute in a unique way the post-impact velocity. The set of ICR is closely related to the dynamical behavior of the regularized system and therefore it represents the propagation phenomena of impulses, but it remains finite in the rigid limit. Finally, it can be computed by a regularized model based on physical observations. It's noteworthy that this computation can be made off-line and leads necessarily to a set of post-impact velocities in agreement with unilateral constraints.

References

[1] M. Abadie. Dynamic simulation of rigid bodies: Modelling of frictional contact. In B. Brogliato, editor, *Impacts in Mechanical Systems: Analysis and Modelling*, volume 551 of *LNP*, pages 61–144. Springer, 2000.

[2] P. Ballard. The dynamics of discrete mechanical systems with perfect unilateral constraints. *Archives for Rational Mechanics and Analysis*, 154:199–274, 2000.

[3] B. Brogliato. *Nonsmooth Mechanics: Models, Dynamics and Control*. Communications and Control Engineering. Springer-Verlag, second edition, 1999.

[4] V. Ceanga and Y. Hurmuzlu. A new look to an old problem : Newton's cradle. *Journal of Applied Mechanics, Transactions of A.S.M.E*, 68(4):575–583, July 2001.

[5] E. Falcon, A. Laroche, and C. Coste. Collision of a 1-d column of beads with a wall. *The European Physical Journal B*, 5:111–131, 1998.

[6] M. Frémond. *Non-Smooth Thermo-mechanics*. Springer-Verlag, 2002.

[7] C. Glocker. *Set-Valued Force Laws: Dynamics of Non-Smooth systems*, volume 1 of *Lecture notes in applied mechanics*. Spring Verlag, 2001.

[8] C. Glocker and F. Pfeiffer. Multiple impacts with friction in rigid multibody systems. *Nonlinear Dynamics, Transactions of A.S.M.E.*, 7(4):471–497, 1995.

[9] I. Han and B.J. Gilmore. Multi-body impact motion with friction-analysis, simulation, and experimental validation. *A.S.M.E. Journal of mechanical Design*, 115:412–422, 1993.

[10] F. Herrmann and M. Seitz. How does the ball-chain work ? *American Journal of Physics*, 50(11):977–981, 1982.

[11] J.M. Hertzsch, F. Spahn, and N.V. Brilliantov. On low-velocity collisions of viscoelastic particles. *Journal de Physique II (France)*, 5:1725–1738, 1995.

[12] J.J. Moreau. Unilateral contact and dry friction in finite freedom dynamics. In J.J. Moreau and P.D. Panagiotopoulos, editors, *Nonsmooth mechanics and applications*, number 302 in CISM, Courses ans lectures, pages 1–82. Springer Verlag, 1988.

[13] R. Ramirez, T. Pöschel, N. V. Brilliantov, and T. Schwager. Coefficient of restitution of colliding viscoelastic spheres. *Physical Review E*, 60(4):4465–4472, 1999.

Appendix A. — Multiple impact law for 3-ball chains with dissipative behavior

Let us consider the dynamical system of three rigid balls of equal mass m, described by their center of mass positions q_1, q_2, q_3 and velocities v_1, v_2, v_3. At the instant of multiple impacts, we can write for $v_2 = 0$

$$\begin{cases} m(v_1^+ - v_1) = -p_1 \\ m\, v_2^+ = p_1 - p_2 \\ m(v_3^+ - v_3) = p_2 \end{cases} \tag{8}$$

The dissipation of energy during the impact is given by a total energetic coefficient e :

$$(v_1^+)^2 + (v_2^+)^2 + (v_3^+)^2 \quad = \quad e\,(v_1^2 + v_3^2) \tag{9}$$

and the ICR ratio is defined by $\alpha = p_2/p_1$.

The perfect unilateral constraints requires that

$$v_1^+ \le v_2^+ \le v_3^+ \tag{10}$$

$$p_1 \ge 0, p_2 \ge 0 \tag{11}$$

For a given value of e and α, and particular initial velocities $v_1 = 1, v_2 = v_3 = 0$ the solution of the system at the instant of impact with respect unilateral constraints is given by :

$$p_1 \quad = \quad \frac{(1+\sqrt{\Delta})\,m}{2(1-\;\alpha+\;\alpha^2)} \tag{12}$$

$$p_2 \quad = \quad \frac{(1+\sqrt{\Delta})\,m\,\alpha}{2\,(1-\alpha+\alpha^2)} \tag{13}$$

$$v_1^+ \quad = \quad -\frac{1+\sqrt{\Delta}}{2\,(1-\alpha+\alpha^2)} + 1 \tag{14}$$

$$v_2^+ \quad = \quad -\frac{(1+\sqrt{\Delta})\,(-1+\alpha)}{2\,(1-\alpha+\alpha^2)} \tag{15}$$

$$v_3^+ \quad = \quad \frac{(1+\sqrt{\Delta})\,\alpha}{2\,(1-\alpha+\alpha^2)} \tag{16}$$

$$\text{with } \Delta \quad = \quad -1 + 2\,e - 2\,\alpha\,e + 2\,\alpha + 2\,\alpha^2\,e - 2\,\alpha^2 \tag{17}$$

This solution requires that Δ is a positive value. With the unilateral constraints, this remark leads to the following inequalities :

$$\frac{1}{3} \quad \le e \le \quad 1 \tag{18}$$

$$\frac{1}{2} \quad \le \alpha \le \quad \frac{2\,e - 2 - \sqrt{6\,e-2}}{(e-3)} \tag{19}$$

The first inequality, which is a very usual result shows that the maximal energy which can be dissipated in impact is bounded. It's noteworthy that both inequalities are correctly correlated with numerical experiments with regularized Hertz model. For a linear spring model in the conservative case (e=1), we can prove this bound analytically.

We give on Figure 3 the values of the post-impact velocities and the impulses for the conservative case with the respect to α. We can see that "ideal" solutions $[v_1^+, v_2^+, v_3^+] = [0, 0, 1]$ and $[v_1^+, v_2^+, v_3^+] = [-1/3, 2/3, 2/3]$ can be found for extremal values of α.

Finally, we given the solution for a general initial velocity set :

$$
\begin{aligned}
p_1 &= \frac{(v_1 - v_2) - t(v_3 - v_2) + \sqrt{(\Delta)}}{2(1 - t + t^2)} \qquad (20)\\
\text{with } \Delta &= 2t^2ev_1^2 - 2tev_3^2 - t^2v_2^2 - t^2v_3^2 + 2t^2ev_2^2 - v_1^2 - v_2^2 - 2v_3^2 + 2ev_3^2 + 2ev_1^2\\
&+ 2ev_2^2 + 2t^2ev_3^2 - 2v_2v_1 - 2t^2v_3v_2 + 2v_2tv_3 - 2v_1tv_3 + 2v_1tv_2 - 2tev_1^2\\
&- 2tev_2^2 + 2tv_3^2 + 2tv_1^2 - 2t^2v_1^2 \qquad (21)
\end{aligned}
$$

Other unknowns can be found in substituting (20) in (8).

Appendix B. — Analytical results for regularized 3-ball chain with linear springs

Let us now analyze the 3-ball chain with linear springs. This model is useful if we want to perform some analytical developments which are intractable with the Hertz model.

For example, let us consider, $v_1 > 0, v_2 = v_3 = 0$ with $\kappa > 1$. We can demonstrate that there exists a non-zero interval $[0, t^\star]$ in which the system behaves as the following bilateral system :

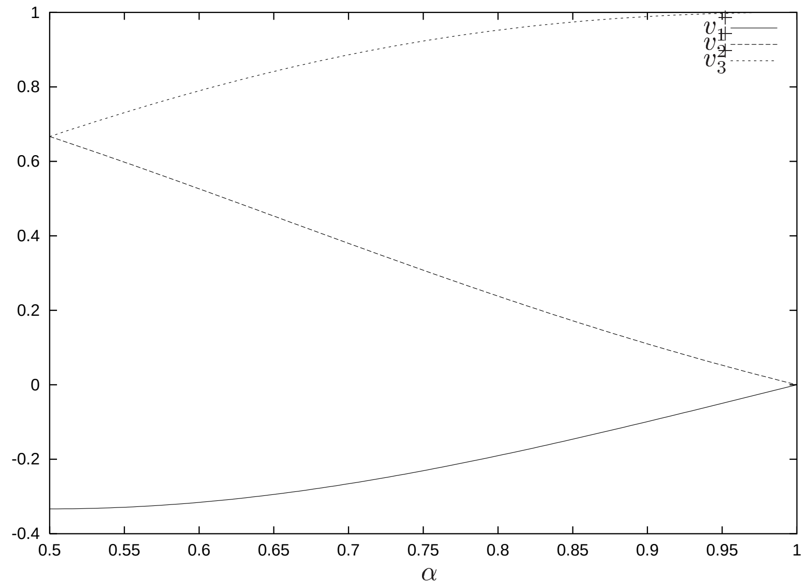
$$
\begin{cases}
m\ddot{\delta}_1 = -2k\delta_1 + \kappa k\delta_2\\
m\ddot{\delta}_2 = -2\kappa k\delta_2 + k\delta_1\\
\delta_1(0) = \delta_2(0) = 0, \quad \dot{\delta}_1(0) = -v_1, \quad \dot{\delta}_2(0) = v_3
\end{cases} \qquad (22)
$$

On $[0, t^\star]$, the solution of (22) is:

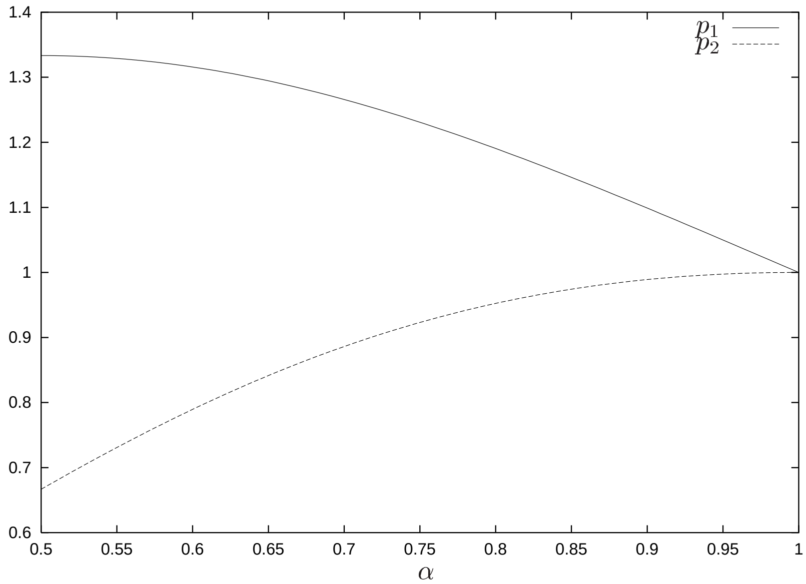
$$
\begin{cases}
\delta_1(t) = \dfrac{-v_1}{\beta - \gamma}\left(\dfrac{\beta}{\omega_1}\sin(\omega_1 t) - \dfrac{\gamma}{\omega_2}\sin(\omega_2 t)\right)\\
\delta_2(t) = \dfrac{-\beta\gamma v_1}{\beta - \gamma}\left(\dfrac{1}{\omega_2}\sin(\omega_2 t) - \dfrac{1}{\omega_1}\sin(\omega_1 t)\right)
\end{cases} \qquad (23)
$$

where (ω_i, ϕ_i) are the natural modes of the system given by :

$$
\begin{cases}
\omega_1^2 = \dfrac{k}{m}\left(\kappa + 1 - \sqrt{\kappa^2 - \kappa + 1}\right), \quad \phi_1 = [\beta = \kappa - 1 + \sqrt{\kappa^2 - \kappa + 1}, 1]^T\\
\omega_2^2 = \dfrac{k}{m}\left(\kappa + 1 + \sqrt{\kappa^2 - \kappa + 1}\right), \quad \phi_2 = [\gamma = \kappa - 1 - \sqrt{\kappa^2 - \kappa + 1}, 1]^T
\end{cases} \qquad (24)
$$



(a) Post-impact velocities



(b) Impulses

Figure 3: Post-impact velocities and the impulses for the conservative case with the respect to α

The first time one contact breaks, denoted as $t^\star$, is provided by the smallest positive root of the transcendental equations:

$$
\begin{aligned}
t_{12} &= \min_{t \in \mathbb{R}^{+*}} \left\{ f_1(t) = 0 \text{ with } f_1(t) = \sin(\omega_1 t) - \frac{\omega_1}{\omega_2}\frac{\gamma}{\beta}\sin(\omega_2 t) \right\} \text{ (first pair of balls)} \\
t_{23} &= \min_{t \in \mathbb{R}^{+*}} \left\{ f_2(t) = 0 \text{ with } f_2(t) = \sin(\omega_1 t) - \frac{\omega_1}{\omega_2}\sin(\omega_2 t) \right\} \text{ (second pair of balls)}
\end{aligned}
$$

Finding the smallest root with respect to the physical parameters of the system is a painful work. However, for this particular case , the following holds:

Proposition 4.1
If $\omega_2/\omega_1 = j \in \mathbb{N}^*$ then $t_{12} = t_{23} = t^\star = \pi/\omega_1$.
If $\omega_2/\omega_1 \in (j; j+1), j \in \mathbb{N}^*$ and j odd (resp. even) then $t^\star = t_{12} < t_{23}$ (resp. $t_{12} > t_{23} = t^\star$).

For $t > t^\star$, only two balls are still in contact. The rest of the process is easily integrable up to the final separation at the time t_f. Moreover, one can show that there is no further contact between the balls as illustrated in Figure 1(e).

For $t^\star = t_{12} < t_{23}$, the ICR is calculated as follows:

$$
\begin{aligned}
\alpha = \frac{p_1}{p_2} &= \frac{1}{\beta\gamma}\left(\frac{-\beta}{\omega_1^2}\left[\cos(\omega_1 t_{12}) - 1\right] - \frac{-\gamma}{\omega_2^2}\left[\cos(\omega_2 t_{12}) - 1\right]\right) / \\
&\quad \left[\frac{1}{\omega_2^2}\left(\cos(\omega_2 t_{12}) - 1\right) - \frac{1}{\omega_1^2}\left(\cos(\omega_1 t_{12}) - 1\right)\right. \\
&\qquad + \frac{1}{{\omega_2'}^2}\left(\cos(\omega_2 t_{12}) - \cos(\omega_1 t_{12})\right)\left(\cos(\omega_2' \hat{t}_{23}) - 1\right) \\
&\qquad\quad \left. - \frac{1}{\omega_2'}\left(\frac{1}{\omega_2}\sin(\omega_2 t_{12}) - \frac{1}{\omega_1}\sin(\omega_1 t_{12})\right)\left(\sin(\omega_2' \hat{t}_{23})\right)\right]
\end{aligned}
\tag{25}
$$

where $\omega_2' = \sqrt{2\kappa k/m}$ is the natural pulsation of two balls in contact and $\hat{t}_{23} = t_{23} - t^\star$.