Lecture 1. Formulations of Non Smooth Dynamical Systems (NSDS).

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The smooth multibody dynamics

Definition (Lagrange's equations)

$$\frac{d}{dt}\left(\frac{\partial L(q,v)}{\partial v_i}\right) - \frac{\partial L(q,v)}{\partial q_i} = Q_i(q,t), \quad i \in \{1\dots n\}, \tag{1}$$

where

- $q(t) \in \mathbb{R}^n$ generalized coordinates,
- $v(t) = \dfrac{dq(t)}{dt} \in \mathbb{R}^n$ generalized velocities,
- $Q(q,t) \in \mathbb{R}^n$ generalized forces
- $L(q,v) \in \mathbb{R}$ Lagrangian of the system,

$$L(q,v) = T(q,v) - V(q),$$

together with

- $T(q,v) = \frac{1}{2}v^T M(q)v$, kinetic energy, $M(q) \in \mathbb{R}^{n\times n}$ mass matrix,
- $V(q)$ potential energy of the system,

The smooth multibody dynamics

Lagrange equations

$$M(q)\frac{dv}{dt} + N(q,v) = Q(q,t) - \nabla_q V(q) \quad (2)$$

where

- $N(q,v) = \left[\frac{1}{2}\sum_{k,l}\frac{\partial M_{ik}}{\partial q_l} + \frac{\partial M_{il}}{\partial q_k} - \frac{\partial M_{kl}}{\partial q_i}, i = 1 \ldots n\right]$ the nonlinear inertial terms i.e., the gyroscopic accelerations

Internal and external forces which do not derive from a potential

$$M(q)\frac{dv}{dt} + N(q,v) + F_{int}(t,q,v) = F_{ext}(t), \quad (3)$$

where

- $F_{int} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ non linear interactions between bodies,
- $F_{ext} : \mathbb{R} \to \mathbb{R}^n$ external applied loads.

Linear time invariant (LTI) case

- $M(q) = M \in \mathbb{R}^{n\times n}$ mass matrix
- $F_{int}(t,q,v) = Cv + Kq$, $C \in \mathbb{R}^{n\times n}$ is the viscosity matrix, $K \in \mathbb{R}^{n\times n}$

The smooth multibody dynamics

Definition (Smooth multibody dynamics)

$$\begin{cases} M(q)\dfrac{dv}{dt} + F(t,q,v) = 0, \\ v = \dot{q} \end{cases} \tag{4}$$

where

- $F(t,q,v) = N(q,v) + F_{int}(t,q,v) - F_{ext}(t)$

Definition (Boundary conditions)

- Initial Value Problem (IVP):

$$t_0 \in \mathbb{R}, \quad q(t_0) = q_0 \in \mathbb{R}^n, \quad v(t_0) = v_0 \in \mathbb{R}^n, \tag{5}$$

- Boundary Value Problem (BVP):

$$(t_0, T) \in \mathbb{R} \times \mathbb{R}, \quad \Gamma(q(t_0), v(t_0), q(T), v(T)) = 0 \tag{6}$$

Perfect bilateral constraints, joints, liaisons and spatial boundary conditions

Bilateral constraints

- Finite set of m bilateral constraints on the generalized coordinates :

$$h(q,t) = \left[h_j(q,t) = 0, \quad j \in \{1 \dots m\}\right]^T. \tag{7}$$

  where h_j are sufficiently smooth with regular gradients, $\nabla_q(h_j)$.
- Configuration manifold, $\mathcal{M}(t)$

$$\mathcal{M}(t) = \{q(t) \in \mathbb{R}^n, h(q,t) = 0\}, \tag{8}$$

Tangent and normal space

- Tangent space to the manifold $\mathcal{M}$ at q

$$T_{\mathcal{M}}(q) = \{\xi, \nabla h(q)^T \xi = 0\} \tag{9}$$

- Normal space as the orthogonal to the tangent space

$$N_{\mathcal{M}}(q) = \{\eta, \eta^T \xi = 0, \forall \xi \in T_{\mathcal{M}}\} \tag{10}$$

Bilateral constraints as inclusion

Definition (Perfect bilateral holonomic constraints on the smooth dynamics)

Introduction of the multipliers $\mu \in \mathbb{R}^m$

$$\begin{cases} M(q)\dfrac{dv}{dt} + F(t,q,v) = r = \nabla_q^T h(q,t)\,\mu \\ \\ -r \in N_{\mathcal{M}}(q) \end{cases} \tag{11}$$

where $r = \nabla_q^T h(q,t)\,\mu$ generalized forces or generalized reactions due to the constraints.

Remark

- The formulation as an inclusion is very useful in practice
- The constraints are said to be perfect due to the normality condition.

Perfect unilateral constraints

Unilateral constraints

- Finite set of ν unilateral constraints on the generalized coordinates :

$$g(q,t) = [g_\alpha(q,t) \geqslant 0, \quad \alpha \in \{1 \dots \nu\}]^T. \tag{12}$$

- Admissible set $\mathcal{C}(t)$

$$\mathcal{C}(t) = \{q \in \mathcal{M}(t), g_\alpha(q,t) \geqslant 0, \alpha \in \{1 \dots \nu\}\}. \tag{13}$$

Normal cone to $\mathcal{C}(t)$

$$N_{\mathcal{C}(t)}(q(t)) = \left\{ y \in \mathbb{R}^n, y = -\sum_\alpha \lambda_\alpha \nabla g_\alpha(q,t), \ \lambda_\alpha \geqslant 0, \ \lambda_\alpha g_\alpha(q,t) = 0 \right\} \tag{14}$$

Unilateral constraints as an inclusion

Definition (Perfect unilateral constraints on the smooth dynamics)

Introduction of the multipliers $\mu \in \mathbb{R}^m$

$$\begin{cases} M(q)\dfrac{dv}{dt} + F(t,q,v) = r = \nabla_q^T h(q,t)\,\lambda \\ \\ -r \in N_{\mathcal{C}(t)}(q(t)) \end{cases} \tag{15}$$

where $r = \nabla_q^T g(q,t)\,\lambda$ generalized forces or generalized reactions due to the constraints.

Remark

- The unilateral constraints are said to be perfect due to the normality condition.
- Notion of normal cones can be extended to more general sets. see (Clarke, 1975, 1983 ; Mordukhovich, 1994)
- The right hand side is neither bounded (and then nor compact).
- The inclusion and the constraints concern the second order time derivative of q.

➜ Standard Analysis of DI does no longer apply.

Non Smooth Lagrangian Dynamics

Fundamental assumptions.

- The velocity $v = \dot{q}$ is of Bounded Variations (B.V)
  ➜ The equation are written in terms of a right continuous B.V. (R.C.B.V.) function, v^+ such that

$$v^+ = \dot{q}^+ \tag{16}$$

- q is related to this velocity by

$$q(t) = q(t_0) + \int_{t_0}^{t} v^+(t)\, dt \tag{17}$$

- The acceleration, ($\ddot{q}$ in the usual sense) is hence a differential measure dv associated with v such that

$$dv(]a,b]) = \int_{]a,b]} dv = v^+(b) - v^+(a) \tag{18}$$

Non Smooth Lagrangian Dynamics

Definition (Non Smooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = dr \\ v^+ = \dot{q}^+ \end{cases} \qquad (19)$$

where dr is the reaction measure and dt is the Lebesgue measure.

Remarks

- The non smooth Dynamics contains the impact equations and the smooth evolution in a single equation.
- The formulation allows one to take into account very complex behaviors, especially, finite accumulation (Zeno-state).
- This formulation is sound from a mathematical Analysis point of view.

References

(Schatzman, 1973, 1978 ; Moreau, 1983, 1988)

Decomposition of measure

$$\left\{\begin{array}{rrrr} dv = & \gamma\, dt + & (v^+ - v^-)\, d\nu + & dv_s \\ dr = & f\, dt + & p\, d\nu + & dr_s \end{array}\right. \tag{20}$$

where

- $\gamma = \ddot{q}$ is the acceleration defined in the usual sense.
- f is the Lebesgue measurable force,
- $v^+ - v^-$ is the difference between the right continuous and the left continuous functions associated with the B.V. function $v = \dot{q}$,
- $d\nu$ is a purely atomic measure concentrated at the time t_i of discontinuities of v, i.e. where $(v^+ - v^-) \neq 0$,i.e. $d\nu = \sum_i \delta_{t_i}$
- p is the purely atomic impact percussions such that $pd\nu = \sum_i p_i \delta_{t_i}$
- dv_S and dr_S are singular measures with the respect to $dt + d\eta$.

Substituting the decomposition of measures into the non smooth Lagrangian Dynamics, one obtains

Definition (Impact equations)

$$M(q)(v^+ - v^-)d\nu = p d\nu, \tag{21}$$

or

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{22}$$

Definition (Smooth Dynamics between impacts)

$$M(q)\gamma dt + F(t, q, v)dt = f dt \tag{23}$$

or

$$M(q)\gamma^+ + F(t, q, v^+) \quad = \quad f^+ \quad [dt - a.e.] \tag{24}$$

The Moreau's sweeping process of second order

Definition (Moreau (1983, 1988))

A key stone of this formulation is the inclusion in terms of velocity. Indeed, the inclusion (15) is "replaced" by the following inclusion

$$\begin{cases} M(q)dv + F(t, q, v^+)dt = dr \\ v^+ = \dot{q}^+ \\ -dr \in N_{T_C(q)}(v^+) \end{cases} \tag{25}$$

Comments

This formulation provides a common framework for the non smooth dynamics containing inelastic impacts without decomposition.
➜ Foundation for the time–stepping approaches.

The Moreau's sweeping process of second order

Comments

- *The inclusion concerns measures.* Therefore, it is necessary to define what is the inclusion of a measure into a cone.
- *The inclusion in terms of velocity v^+* rather than of the coordinates q.

Interpretation

- Inclusion of measure, $-dr \in K$
  - Case $dr = r'dt = fdt$.
    $$-f \in K \tag{26}$$
  - Case $dr = p_i\delta_i$.
    $$-p_i \in K \tag{27}$$
- Inclusion in terms of the velocity. Viability Lemma
  If $q(t_0) \in C(t_0)$, then
  $$v^+ \in T_C(q), t \geqslant t_0 \Rightarrow q(t) \in C(t), t \geqslant t_0$$
  ➜ The unilateral constraints on q are satisfied. The equivalence needs at least an impact inelastic rule.

The Moreau's sweeping process of second order

The Newton-Moreau impact rule

$$-dr \in N_{T_C(q(t))}(v^+(t) + ev^-(t)) \tag{28}$$

where e is a coefficient of restitution.

The Moreau's sweeping process of second order

The case of C is finitely represented

$$\mathcal{C} = \{q \in \mathcal{M}(t), g_\alpha(q) \geqslant 0, \alpha \in \{1 \ldots \nu\}\}. \tag{29}$$

Decomposition of dr and v^+ onto the tangent and the normal cone.

$$dr = \sum_\alpha \nabla_q^T g_\alpha(q)\, d\lambda_\alpha \tag{30}$$

$$U_\alpha^+ = \nabla_q g_\alpha(q)\, v^+, \alpha \in \{1 \ldots \nu\} \tag{31}$$

Complementarity formulation (under constraints qualification condition)

$$-d\lambda_\alpha \in N_{T_{\mathbb{R}_+}(g_\alpha)}(U_\alpha^+) \Leftrightarrow \text{ if } g_\alpha(q) \leqslant 0, \text{ then } 0 \leqslant U_\alpha^+ \perp d\lambda_\alpha \geqslant 0 \tag{32}$$

The case of C is $\mathbb{R}_+$

$$-dr \in N_C(q) \Leftrightarrow 0 \leqslant q \perp dr \geqslant 0 \tag{33}$$

is replaced by

$$-dr \in N_{T_C(q)}(v^+) \Leftrightarrow \text{ if } q \leqslant 0, \text{ then } 0 \leqslant v^+ \perp dr \geqslant 0 \tag{34}$$

The Moreau's sweeping process of second order

Example (The Bouncing Ball)

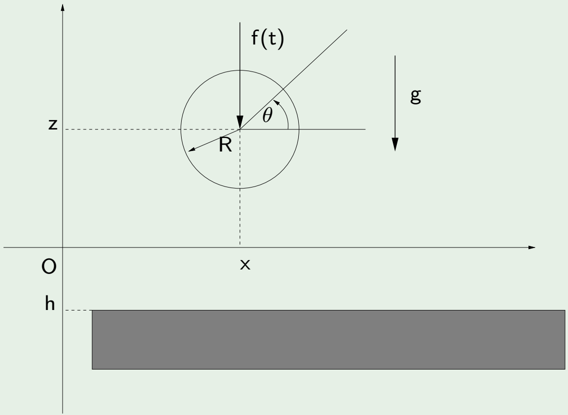

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Figure: Two-dimensional bouncing ball on a rigid plane

Example (The Bouncing Ball)

In our special case, the model is completely linear:

$$q = \begin{bmatrix} z \\ x \\ \theta \end{bmatrix} \tag{35}$$

$$M(q) = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix} \text{ where } I = \frac{3}{5}mR^2 \tag{36}$$

$$N(q,\dot{q}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{37}$$

$$F_{int}(q,\dot{q},t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{38}$$

$$F_{ext}(t) = \begin{bmatrix} -mg \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} f(t) \\ 0 \\ 0 \end{bmatrix} \tag{39}$$

Example (The Bouncing Ball)

Kinematics Relations The unilateral constraint requires that :

$$C = \{q, g(q) = z - R - h \geqslant 0\} \tag{35}$$

so we identify the terms of the equation the equation (30)

$$-\,dr \quad = \quad [1, 0, 0]^T d\lambda_1, \tag{36}$$

$$U_1^+ \quad = \quad [1, 0, 0] \begin{bmatrix} \dot{z} \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \dot{z} \tag{37}$$

Nonsmooth laws The following contact laws can be written,

$$\begin{cases} \text{if } g(q) \leqslant 0, \text{ then } 0 \leqslant U^+ + eU^- \perp d\lambda_1 \geqslant 0 \\ \\ \text{if } g(q) \geqslant 0, d\lambda_1 = 0 \end{cases} \tag{38}$$

The Moreau's sweeping process of first order

Definition (The Moreau's sweeping process (of first order))

The Moreau's sweeping process (of first order) is defined by the following Differential inclusion (DI)

$$\begin{cases} -\dot{x}(t) \in N_{K(t)}(x(t)) \quad t \in [0,T], \\ x(0) = x_0 \in K(0). \end{cases} \tag{39}$$

where

- $K(t)$ is a moving closed and nonempty convex set.
- $N_K(x)$ is the normal cone to K at x

$$N_K(x) := \{s \in \mathbb{R}^n : \langle s, y - x\rangle \leqslant 0, \text{ for all } y \in K\},$$

Comment

This terminology is explained by the fact that $x(t)$ can be viewed as a point which is swept by a moving convex set.

References

(Moreau, 1971, 1972, 1977 ; Monteiro Marques, 1993 ; Kunze & Monteiro Marqus, 2000)

The Moreau's sweeping process of first order

Basic mathematical properties (Monteiro Marques, 1993).

- A solution $x(.)$ for such type of DI is assumed to be differentiable almost everywhere satisfying the inclusion $x(t) \in K(t), t \in [0, T]$.
- If the set-valued application $t \mapsto K(t)$ is supposed to be Lipschitz continuous, i.e.

$$\exists l \leqslant 0, \quad d_H(K(t), K(s)) \leqslant l|t-s| \tag{40}$$

where d_H is the Hausdorff distance between two closed sets, then
  - existence of a solution which is l-Lipschitz continuous
  - uniqueness in the class of absolutely continuous functions.

(Monteiro Marques, 1993).

Definition (State dependent sweeping process (Kunze & Monteiro Marques, 1998))

The state dependent sweeping process is defined

$$\begin{cases} -\dot{x}(t) \in N_{K(t,x(t))}(x(t)) \quad t \in [0, T], \\ x(0) = x_0 \in K(0). \end{cases} \tag{41}$$

Variants of the Moreau's sweeping process

Definition (RCBV sweeping process (Kunze & Monteiro Marques, 1998))

The RCBV sweeping process of the type is defined

$$\begin{cases} -du \in N_{K(t)}(u(t)) \;\; (t \geqslant 0), \\ u(0) = u_0. \end{cases} \tag{42}$$

where the convex set is RCBV i.e

$$d_H(K(t), K(s)) \leqslant r(t) - r(s) \tag{43}$$

for some right-continuous non-decreasing function $r : [0, T] \to \mathbb{R}$ is made.

Mathematical properties

- the solution $u(.)$ is searched as a function of bounded variations (B.V.)
- the measure du associated with the B.V. function u is a differential measure or a Stieltjes measure.
- Inclusion of measure into cone

Unbounded DI and Maximal monotone operator

Definition (Unbounded Differential Inclusion (UDI))

The following UDI can be defined (together with the initial condition $x(0) = x_0 \in C$)

$$-(\dot{x}(t) + f(x(t)) + g(t)) \in \mathbb{N}_K(x(t)) \tag{44}$$

where K is the feasible set and $g : \mathbb{R}_+ \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$.

Basic properties

- A solution of such a UDI is understood as an absolutely continuous $t \mapsto x(t)$ lying in the convex set C.

Comment

The Terminology is explained by the fact that $\mathbb{N}_K(x(t))$ is neither compact nor bounded. Standard DI analysis no longer apply.

Unbounded DI and Maximal monotone operator

Link with Maximal monotone operator

- In (Brezis, 1973), a existence and uniqueness theorem for

$$\dot{x}(t) + A(x(t)) + g(t) \ni 0 \tag{45}$$

  where A is a maximal monotone operator, and g a absolutely continuous function of time.
- If f which is monotone and Lipschitz continuous, then

$$A(x(t)) = f(x(t)) + \mathbb{N}_K(x(t)) \tag{46}$$

  is then a maximal monotone operator.
- Equivalence (Brogliato *et al.*, 2006)

$$-\left(\dot{x}(t) + f(x(t)) + g(t)\right) \in \mathbb{N}_{\mathrm{T}_K(x(t))}(\dot{x}(t)), \tag{47}$$

  providing that the UDI (44) has the so-called slow solution, that is $\dot{x}(t)$ is of minimal norm in $\mathbb{N}_{K(x(t))}(x(t)) + f(x,t) + g(t)$.

Special case when K is finitely represented.

Assumptions

$$K = \{x \in \mathbb{R}^n, h(x) \leqslant 0\} \tag{48}$$

For $x \in K$, we denote by

$$I(x) = \{i \in \{i \dots m\}, h_i(x) = 0\} \tag{49}$$

the set of active constraints at x. The tangent cone can be defined by

$$T^h(x) = \{s \in \mathbb{R}^n, \langle \nabla h_i(x), s\rangle \leqslant 0, i \in I(x)\} \tag{50}$$

and the normal cone by

$$N^h(x) := [T^h(x)]^\circ = \{ \sum_{i \in I(x)} \lambda_i \nabla h_i(x), \lambda_i \geqslant 0,\ i \in I(x)\} \tag{51}$$

- $N_K(x) \supset N^h(x)$ and $T_K(x) \subset T^h(x)$ always hold.
- $N_K = N^h$ and equivalently $T_K = T^h$ holds if a constraints qualification condition is satisfied

Special case when K is finitely represented.

Link with Differential Complementarity Systems (DCS)

Equivalence with the following DCS of Gradient Type (GTCS)

$$\begin{cases} -\dot{x}(t) = f(x(t)) + g(t) + \nabla h(x(t))\lambda(t) \\ 0 \leqslant -h(x(t)) \perp \lambda(t) \geqslant 0 \end{cases} \tag{48}$$

Link with Evolution Variational Inequalities (EVI)

Equivalence with the following EVI

$$\langle \dot{x}(t) + f(x(t)) + g(t), y - x \rangle \geqslant 0 \tag{49}$$

- existence and uniqueness theorem for maximal monotone operators
- existence result is given for this last EVI under the assumption that f is continuous and hypo-monotone (Brogliato *et al.*, 2006).

Applications

- Quasi-static analysis (first order) of viscoelastic mechanical systems
  - with perfect (associated) plasticity
  - with associated friction
- Quasi static analysis (first order) of quasi-brittle mechanical systems
  - cohesion, damage and fracture mechanics
  - geomaterials

1 Outline

2 Lagrangian dynamical systems with unilateral constraints
- The smooth multibody dynamics
- The Non smooth Lagrangian Dynamics
- The Moreau's sweeping process

3 The Moreau's sweeping process of first order

4 Dynamical Complementarity Systems (DCS)
- Definitions
- The notion of relative degree. Well-posedness
- The LCS of relative degree $r \leqslant 1$. The passive LCS

5 Other Non Smooth Dynamical systems: A very short zoology
- Differential inclusions (DI)
- Evolution Variational inequalities (EVI)
- Differential Variational Inequalities (DVI)
- Projected Dynamical Systems (PDS)
- Piece-Wise affine (PWA) and piece-wise continuous (PWC) systems
- And other systems

6 Higher order relative degree systems

Dynamical Complementarity systems

Definition (Generalized Dynamical Complementarity Systems (GDCS) (semi-explicit form))

A generalized Dynamical Complementarity System (DCS) in a semi-explicit form is defined by

$$\begin{cases} \dot{x} = f(x, t, \lambda) \\ y = h(x, \lambda) \\ C^* \ni y \perp \lambda \in C \end{cases} \tag{50}$$

where C and C^* are a pair of dual closed convex cones ($C^* = -C^\circ$).

Definition (Dynamical Complementarity Systems (DCS))

A Dynamical Complementarity System (DCS) in a explicit form is defined by

$$\begin{cases} \dot{x} = f(x, t, \lambda) \\ y = h(x, \lambda) \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{51}$$

Dynamical Complementarity systems

Definition (Linear Complementarity Systems (LCS))

A Linear Complementarity System (LCS) is defined by

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{50}$$

Dynamical Complementarity systems

Definition (Non Linear complementarity systems (NLCS))

A Non Linear Complementarity System usually (NLCS) is defined by the following system:

$$\begin{cases} \dot{x} = f(x,t) + g(x)^T \lambda \\ y = h(x,\lambda) \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{50}$$

Definition (Gradient Type Complementarity Problem (GTCS))

A Gradient Type Complementarity Problem (GTCS) is defined by the following system:

$$\begin{cases} \dot{x}(t) + f(x(t)) = \nabla_x^T g(x)\lambda \\ y = g(x(t)) \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{51}$$

The notion of relative degree. Well-posedness

Definition (Relative degree in the SISO case)

Let us consider a linear system in state representation given by the quadruplet $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times m}$:

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \end{cases} \tag{52}$$

- In the Single Input/ Single Output (SISO) case ($m = 1$), the relative degree is defined by the first non zero Markov parameters :

$$D, CB, CAB, CA^2B, \ldots, CA^{r-1}B, \ldots \tag{53}$$

- In the multiple input/multiple output (MIMO) case ($m > 1$), an *uniform* relative degree is defined as follows. If D is non singular, the relative degree is equal to 0. Otherwise, it is assumed to be the first positive integer r such that

$$CA^iB = 0, \quad i = 0 \ldots q-2 \tag{54}$$

while

$$CA^{r-1}B \text{ is non singular.} \tag{55}$$

The notion of relative degree. Well-posedness

Interpretation

The Markov parameters arise naturally when we derive with respect to time the output y,

$$\begin{array}{rcl} y & = & Cx + D\lambda \\ \dot{y} & = & CAx + CB\lambda, \text{ if } D = 0 \\ \ddot{y} & = & CA^2x + CAB\lambda, \text{ if } D = 0, CB = 0 \\ & \dots & \\ y^{(r)} & = & CA^rx + CA^{r-1}B\lambda, \text{ if } D = 0, CB = 0, CA^{r-2}B = 0, r = 1 \dots r-2 \\ & \dots & \end{array}$$

and the first non zero Markov parameter allows us to define the output y directly in terms of the input λ.

The notion of relative degree. Well-posedness

Example

Third relative degree LCS Let us consider the following LCS:

$$\begin{cases} \dddot{x}(t) = \lambda, x(0) = x_0 \geqslant 0 \\ y(t) = x(t) \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{52}$$

The function $x : [0, T] \to \mathbb{R}$ is usually assumed to be an absolutely continuous function of time.

- If $y = x \geqslant 0$ becomes active, i.e., $x = 0$,
  - If $\dot{x} > 0$, the system will instantaneously leaves the constraints.
  - If $\dot{x} < 0, \ddot{x} > 0$, the velocity needs to jump to respect the constraint in t^+. (B.V. function ?)
  - If $\dot{x} < 0, \ddot{x} < 0$, the velocity and the acceleration need to jump to respect the constraint in t^+. (Dirac + B.V. function)

➜ $\ddot{x} < 0$ and therefore λ may be derivative of Dirac distribution.

Problem: From the mathematical point of view, a constraint of the type $\lambda \geqslant 0$ has no mathematical meaning !!

Restrictions

➜ In this lecture, we will focus on LCS of relative degree $r \leqslant 1$.

The passive LCS.

Relative degree 0

Let us consider a LCS of relative degree 0 i.e. with D which is non singular.

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x(0) = x_0 \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{53}$$

Mathematical properties

D is non singular ➜ poor interest

- Existence and Uniqueness.
  - "$B.SOL(Cx, D)$ is a singleton":
    $B.SOL(Cx_0, D)$ is a singleton is equivalent to stating that the LCS (57) has a unique C^1 solution defined at all $t \geqslant 0$.
    Denoting by $\Lambda(x) = B.SOL(Cx, D)$, the LCS can be viewed as a standard ODE with a Lipschitz r.h.s :
    $$\dot{x} = Ax + \Lambda(x) = Ax + B.SOL(Cx, D) \tag{54}$$
  - Special important case: D is a P-matrix, ($LCP(q, M)$ has a unique solution for all $q \in \mathbb{R}^n$ if M is a P-matrix.) The Lipschitz property of the LCP solution with the respect to x is shown in Cottle *et al.* (1992).
- Stability theory (Camlibel *et al.*, 2006) and for the numerical integration, the problem is a little more tricky because $\Lambda(x)$ is only B-differentiable.

The passive LCS.

Example

To complete this section, a example of non existence and non uniqueness of solutions is provided for a LCS of relative degree 0. This example is taken from HEEMELS & BROGLIATO (2003). Let us consider the following LCS

$$\begin{cases} \dot{x} = -x + \lambda \\ y = x - \lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{55}$$

This system is strictly equivalent to

$$\dot{x} = \begin{cases} -x, & \text{if } x \geqslant 0 \\ 0, & \text{if } x \geqslant 0 \end{cases} \tag{56}$$

which leads to non existence of solutions for $x(0) < 0$ and to non uniqueness for for $x(0) > 0$.

The passive LCS.

Relative degree 1

Let us consider a LCS of relative degree 1 i.e. with CB which is non singular.

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x(0) = x_0 \\ y = Cx \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{57}$$

Mathematical properties

- The Rational Complementarity problem Heemels (1999) ; Camlibel (2001) ; Camlibel *et al.* (2002). The P-matrix property plays henceforth a fundamental role and provides the existence of global solution of the LCS in the sense of Caratheodory.
- Special case $B = C^T$ uses some EVI results for the well-posedness and the stability of such a systems (Goeleven & Brogliato, 2004).

The passive LCS.

Comments

The passive linear systems are a class for which a "stored energy" in the system is only decreasing (see for more details, (Camlibel, 2001 ; Heemels & Brogliato, 2003)). The passive linear systems are of relative degree $\geqslant 1$.

The passive LCS.

Example (The RLC circuit with a diode)

A LC oscillator supplying a load resistor through a half-wave rectifier (see figure 1).

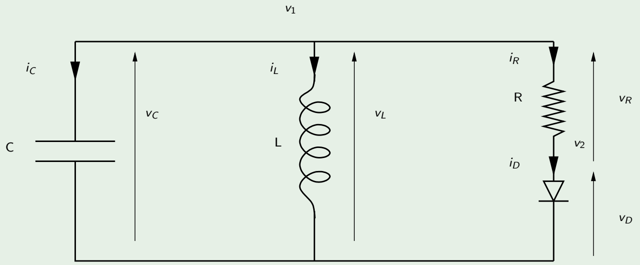

Figure: Electrical oscillator with half-wave rectifier

The passive LCS.

Example (The RLC circuit with a diode)

- Kirchhoff laws :

$$\begin{aligned} v_L &= v_C \\ v_R + v_D &= v_C \\ i_C + i_L + i_R &= 0 \\ i_R &= i_D \end{aligned}$$

- Branch constitutive equations for linear devices are :

$$\begin{aligned} i_C &= C\dot{v}_C \\ v_L &= L\dot{i}_L \\ v_R &= Ri_R \end{aligned}$$

- "branch constitutive equation" of the ideal diode

$$0 \leqslant i_D \perp -v_D \geqslant 0$$

The passive LCS.

Example (The RLC circuit with a diode)

The following LCS is obtained :

$$\begin{pmatrix} \dot{v}_L \\ \dot{i}_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{pmatrix} \cdot \begin{pmatrix} v_L \\ i_L \end{pmatrix} + \begin{pmatrix} \frac{-1}{C} \\ 0 \end{pmatrix} \cdot i_D$$

together with a state variable x and one of the complementary variables λ :

$$x = \begin{pmatrix} v_L \\ i_L \end{pmatrix}$$

and

$$\lambda = i_D$$

Differential inclusions (DI)

Definition

A differential inclusion (DI) may be defined by

$$\dot{x}(t) \in F(x(t)), t \in [0, T] \tag{58}$$

where

- $x(t) : \mathbb{R} \rightarrow \mathbb{R}^n$ is a function of time t,
- $\dot{x}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$ is the time derivative,
- $F : \mathbb{R} \rightarrow \mathbb{R}^n$ is a set-valued map which associates to any point $x \in \mathbb{R}^n$ a set $F(x) \subset \mathbb{R}^n$.

Standard classes of DI

- Lipschitzian DI
- Upper semi-continuous DI

Standard references

(Aubin & Cellina, 1984 ; Deimling, 1992 ; Smirnov, 2002)

Differential inclusions (DI)

Example

Ordinary Differential Equation (ODE)

$$\dot{x} = f(x,t), \tag{59}$$

considering the singleton $F(x) = \{f(x,t)\}$

Example

Implicit Differential Equation (IDE),

$$f(\dot{x}, x) = 0 \tag{60}$$

defining the set-valued map as $F(x) = \{v, f(v,x) = 0\}$

Differential inclusions (DI)

Example

ODE with discontinuous right hand side (r.h.s.),

$$\dot{x}(t) = f(x(t)), t \in [0, T] \tag{61}$$

with

$$f(x,t) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x \geqslant 0 \end{cases} \tag{62}$$

Filippov DI :

$$\dot{x}(t) \in F(x) = \bigcap_{\varepsilon > 0} \overline{\mathrm{conv}} f(x + \varepsilon B_n) \tag{63}$$

where B_n is the unit ball of $\mathbb{R}^n$.

Why DIs are Non Smooth Dynamical systems ?

- Extensive use of Non Smooth and Set-valued Analysis.
- Non smoothness of solution due to constraints on $\dot{x}$
  - $x(t)$ is usually absolutely continuous
  - $\dot{x}(t)$ is usually non smooth ($\mathcal{L}^1$, B.V. functions)

Lipschitzian DI

Definition (Lipschitzian DI)

A DI is said to be a Lipschitzian DI if the set-valued map $F : \mathbb{R} \rightarrow \mathbb{R}^n$ satisfies the following condition:

1. the sets $F(x)$ are closed and convex for all $x \in \mathbb{R}^n$;
2. the set-valued map F is Lipschitzian with a constant l, i.e.

$$\exists l \geqslant 0, \quad F(x_1) \subset F(x_2) + l\|x_1 - x_2\|B_n \tag{64}$$

where B_n is the unit ball of $\mathbb{R}^n$,

Lipschitzian DI

Example (Control theory)

- ODE with control input

$$\dot{x} = f(x,u), t \in [0,T], x(0) = x_0 \qquad u \in U \subset \mathbb{R}^m \tag{65}$$

where $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is assumed to be a continuous function satisfying a Lipschitz condition in x.

- Associated Lipschitzian DI

$$\dot{x} \in \cup_{u \in U} f(x,u) \tag{66}$$

Assume that the set $f(x,U)$ is closed and convex for all $x \in \mathbb{R}^n$, the solution of the Cauchy problem (65) is a solution of the DI (66) and due to a result of Filippov, the converse statement is also true in the sense that there exists a solution $v(t)$ of the inclusion (66) which is also a solution of (65).

Upper semi-continuous DI

Definition (Upper semi-continuous DI)

A DI is said to be an upper semi-continuous DI if the set-valued map $F : \mathbb{R} \rightarrow \mathbb{R}^n$ satisfies the following condition:

1. the sets $F(x)$ are closed and convex for all $x \in \mathbb{R}^n$;
2. the set-valued map F is upper semi-continuous for all $x \in \mathbb{R}$, i.e, if for every open set M containing $F(x), x \in \mathbb{R}$ there exists a neighborhood Ω of x such that $F(\Omega) \subset M$.

An example of upper semi-continuous DI: the Filippov DI

$$\dot{x}(t) = f(x(t)), t \in [0, T], \quad x(0) = x_0 \tag{67}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a bounded function.

If f is not continuous, then the Cauchy problem associated with this ODE may have no solution.

Filippov DI

$$\dot{x}(t) \in F(x) = \bigcap_{\varepsilon > 0} \overline{\mathrm{conv}} f(x + \varepsilon B_n) \tag{68}$$

where B_n is the unit ball of $\mathbb{R}^n$.

Upper semi-continuous DI

Example (ODE with a discontinuous r.h.s)

A standard example is given by the following r.h.s:

$$f(x,t) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x \geqslant 0 \end{cases} \tag{67}$$

Standard solution

$$\begin{cases} x(t) < 0, x(t) = t + x_0 \\ x(t) > 0, x(t) = -t + x_0 \end{cases} \tag{68}$$

Each solution reaches the point $x = 0$ and can not leave it. Unfortunately, the function $x(t) \equiv 0$ does not satisfy the equation, since $\dot{x} = 0 \neq f(0) = -1$.

Filippov DI

$$\dot{x}(t) \in F(x) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x > 0 \\ [-1,1], & \text{if } x = 0 \end{cases} \tag{69}$$

Variational inequalities (VI)

Definition (Variational Inequality (VI) problem)

Let X be a nonempty subset of $\mathbb{R}^n$ and let F be a mapping form $\mathbb{R}^n$ into itself. The Variational Inequality problem, denoted by $\mathrm{VI}(X, F)$ is to find a vector $z \in \mathbb{R}^n$ such that

$$F(z)^T(y - z) \geq 0, \forall y \in X \tag{70}$$

Equivalences and others definitions

- Inclusion into a normal cone.

$$-F(x) \in N_X(x) \tag{71}$$

  or equivalently

$$0 \in F(x) + N_X(x) \tag{72}$$

- If F is affine function, $F(x) = Mz + q$, the $\mathrm{VI}(X, F)$ is called Affine VI denoted by, $\mathrm{AVI}(X, F)$.
- If X is polyhedral, we say that the $\mathrm{VI}(X, F)$ is linearly constrained.

Evolution Variational inequalities (EVI)

Definition (Evolution Variational Inequalities (EVI))

An Evolution Variational Inequality (EVI) is defined by finding $x \in K$ such that

$$\langle \dot{x} + f(x), y - x \rangle \geqslant 0, \forall y \in K \tag{73}$$

which is equivalent to the following unbounded DI

$$-(\dot{x} + f(x)) \in \mathbb{N}_K(x) \tag{74}$$

References

- Infinite-dimensional spaces. (Lions & Stampacchia, 1967 ; Kinderlehrer & Stampacchia, 1980 ; Goeleven *et al.*, 2003)
- Finite-dimensional spaces. (Harker & Pang, 1990 ; Facchinei & Pang, 2003)

Reformulation into a Unbounded DI

Mathematical properties

- Trough the reformulation (44),existence and uniqueness theorem for maximal monotone operators holds for

$$\langle \dot{x}(t) + f(x(t)) + g(t), y - x \rangle \geqslant 0 \tag{75}$$

In (Brogliato *et al.*, 2006), a existence result is given under the assumption that f is continuous and hypo-monotone.

Other definitions

- For $g \equiv 0$ and $f(x) = Ax$, the EVI is called a Linear Evolution Variational Inequality (LEVI).
- If the set K depends on x, i.e. $K(x)$, we speak of Evolution Quasi-Variational inequality (EQVI)

$$\langle \dot{x} + f(x), y - x \rangle \geqslant 0, \forall y \in K(x) \tag{76}$$

Differential Variational Inequalities (DVI)

Definition (Differential Variational inequalities (DVI) (Pang, 2006))

A Differential Variational inequality can be defined as follows:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)) && (77)\\ u(t) &= \mathrm{SOL}(K, F(t, x(t), \cdot)) && (78)\\ 0 &= \Gamma(x(0), x(T)) && (79) \end{aligned}$$

where :

- $x : [0, T] \rightarrow \mathbb{R}^n$ is the differential trajectory (state variable),
- $u : [0, T] \rightarrow \mathbb{R}^m$ is the algebraic trajectory
- $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the ODE right-hand side
- $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the VI function
- K is nonempty closed convex subset of $\mathbb{R}^m$
- $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the boundary conditions function.
  - Initial Value Problem (IVP), $\Gamma(x, y) = x - x_0$
  - linear Boundary Value Problem (BVP), $\Gamma(x, y) = Mx + Ny - b$

The notation $u(t) = \mathrm{SOL}(K, \Phi)$ means that $u(t) \in K$ is the solution of the following VI

$$(v - u)^T \Phi(u) \geqslant 0, \quad \forall v \in K \qquad (80)$$

Differential Variational Inequalities (DVI)

The DVI is a slightly more general framework in the sense that it includes at the same time:

- Differential Algebraic equations(DAE)

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)) && (81) \\ u(t) &= F(t, x(t), u(t)) && (82) \end{aligned}$$

- Differential Complementarity systems (DCS)

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)) && (83) \\ C &\ni u(t) \perp F(t, x(t), u(t)) \in C^* && (84) \end{aligned}$$

where C and C^* are a pair of dual closed convex cones ($C^* = -C^\circ$). The Linear Complementarity systems are also special case of DVI (see the section 4).

Differential Variational Inequalities (DVI)

The DVI is a slightly more general framework in the sense that it includes at the same time:

- Evolution variational inequalities (EVI)

$$-(\dot{x} + f(x)) \in \mathbb{N}_K(x) \quad (81)$$

- When K is a cone, the preceding EVI is equivalent to a DCS of the type :

$$\begin{array}{rcll} \dot{x}(t) + f(x(t)) & = & u(t) & (82) \\ K \ni x(t) \perp u(t) & \in & K^* & (83) \end{array}$$

- When K is finitely represented i.e. $K = \{x \in \mathbb{R}^n, g(x) \leqslant 0\}$ then under some appropriate constraints qualifications, we obtain another DCS which is often called a Gradient type Complementarity Problem (GTCS) (see 4) :

$$\begin{array}{rcll} \dot{x}(t) + f(x(t)) & = & -\nabla_x^T g(x)u(t) & (84) \\ 0 \leqslant -g(x(t)) \perp u(t) & \geqslant & 0 & (85) \end{array}$$

- Finally, if K is a closed convex and nonempty set then the EVI is equivalent to the following DVI :

$$\begin{array}{rcll} \dot{x}(t) + f(x(t)) & = & w(t) & (86) \\ 0 & = & x(t) - y(t) & (87) \\ y(t) \in K,\ (v - y(t))^T w(t) & \geqslant & 0, \forall v \in K & (88) \end{array}$$

Projected Dynamical Systems (PDS)

Definition (Projected Dynamical Systems (PDS))

Let us consider a nonempty closed and convex subset K of $\mathbb{R}^n$. A Projected Dynamical System (PDS) is defined as the following system:

$$\dot{x}(t) = \Pi_K\left(x(t); -(f(x(t)) + g(t))\right) \tag{89}$$

where $\Pi_K : K \times \mathbb{R}^n \to \mathbb{R}^n$ is the operator

$$\Pi_K(x; v) = \lim_{\delta \downarrow 0} \frac{\mathrm{proj}_K(x + \delta v) - x}{\delta} \tag{90}$$

Comments

- The definition of the operator Π_K corresponds to the one-sided Gteaux derivative of the projection operator for $x \in K$, i.e. when $P_K(x) = x$. A classical result of Convex analysis, see for instance (Hirriart-Urruty & Lemarechal, 1993), states that

$$\Pi_K(x; v) = \mathrm{proj}_{T_K(x)}(v) \tag{91}$$

  Therefore, the PDS can be equivalently rewritten as :

$$\dot{x}(t) = \mathrm{proj}_{T_K(x(t))}\left(-(f(x(t)) + g(t))\right) \tag{92}$$

Projected Dynamical Systems (PDS)

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where $\Pi_K : K \times \mathbb{R}^n \to \mathbb{R}^n$ is the operator

$$\Pi_K(x; v) = \lim_{\delta \downarrow 0} \frac{\mathrm{proj}_K(x + \delta v) - x}{\delta} \tag{90}$$

Comments

- In (Brogliato *et al.*, 2006), the PDS (92) is proved to be equivalent to the UDI(47) and therefore to be equivalent to the UDI (44) if the slow condition is selected.
- For results and definitions in infinite-dimensional spaces (Hilbert spaces), we refer to the work of (Cojocaru, 2002 ; Cojocaru & Jonker, 2003).

Piece-Wise affine (PWA) and piece-wise continuous (PWC) systems

Definition (Piece-Wise affine (PWA) systems)

A Piece-Wise affine (PWA) system can be defined by systems of the form

$$\dot{x}(t) = A_i x(t) + a_i, \quad x(t) \in X_i \tag{91}$$

where

- $\{X_i\}_{i\in I} \subset \mathbb{R}^n$, partition of the state space in closed (possibly unbounded) polyhedral cells with disjoint interior,
- the matrix $A_i \in \mathbb{R}^{n\times n}$ and the vector $a_i \in \mathbb{R}^n$ defines an affine system on each cell.

Piece-Wise affine (PWA) and piece-wise continuous (PWC) systems

Nature of solution (JOHANSSON & RANTZER, 1998)

Solution: a continuous piecewise $\mathcal{C}^1$ function $x(t) \in \cup_{i\in I} X_i$ on the time interval $[0, T]$ with for every $t \in [0, T]$ such the derivative $\dot{x}(t)$ is defined, the equation $\dot{x}(t) = A_i x(t) + a_i$, holds for all i with $x(t) \in X_i$..

Remarks

The definition is relatively rough, but can suffice to understand what type of solutions are sought. Indeed, If some discontinuity of the r.h.s is allowed, the canonical problem with the sign function can be cast into such a formalism. We know that the existence of solution is not guaranteed for such a r.h.s. . The authors JOHANSSON & RANTZER (1998) circumvent this problem excluding arbitrarily such cases. A proper definition of solution could be given by the FILIPPOV (1988) or UTKIN (1977) solutions of the system:

$$\dot{x}(t) = \mathrm{conv}_{j\in J}\{A_i x(t) + a_i\} \text{ with } J = \{j, x(t) \in X_j\} \tag{91}$$

Piece-Wise Continuous (PWC) systems

Definition (Piece-Wise Continuous (PWC) systems)

A Piece-Wise Continuous (PWC) systems can be defined by

$$\dot{x}(t) = f_i(x,t), \quad x(t) \in X_i \tag{92}$$

where the continuous $f_i : \mathbb{R}^n \times [0,T] \rightarrow \mathbb{R}^n$ defines an continuous system on each cell.

Comments

In a general way, it is difficult to understand what is the interest in PWA and PWC systems without referring to one of the following formalisms

- ODE with Lipschitz r.h.s
- Filippov DI
- Higher order relative degree systems

And other systems ...

- Time varying systems
- Switched systems
- Hybrid systems
- Impulsive Differential Equations
- ...

Higher order relative degree systems

References

- (Heemels *et al.*, 2000)
- (Acary *et al.*, 2005)

Thank you for your attention.

V. Acary, B. Brogliato & D. Goeleven. Higher order moreau's sweeping process: Mathematical formulation and numerical simulation. *Mathematical Programming A*, 2005.

J.P. Aubin & A. Cellina. *Differential inclusions: set-valued maps and viability theory*. Springer, Berlin, 1984.

H. Brezis. *Oprateurs maximaux monotones et semi-groupes de contraction dans les espaces de Hilbert*. North Holland, Amsterdam, 1973.

B. Brogliato, A. Daniilidis, C. Lemaréchal & V. Acary. On the equivalence between complementarity systems, projected systems and differential inclusions. *Systems and Control Letters*, 55(1), pp. 45–51, 2006.

K. Camlibel. *Complementarity Methods in the Analysis of Piecewise Linear Dynamical Systems*. PhD thesis, Katholieke Universiteit Brabant, 2001. ISBN: 90 5668 073X.

K. Camlibel, W.P.M.H. Heemels & J.M. Schumacher. Consistency of a time-stepping method for a class of piecewise-linear networks. *IEEE Transactions on Circuits and Systems I*, 49, pp. 349–357, 2002.

K. Camlibel, J.S. Pang & J. Shen. Lyaunov stability of complementarity and extended systems. *SIAM Journal on Optimization*, 2006. in revision.

F.H. Clarke. Generalized gradients and its applications. *Transactions of A.M.S.*, 205, pp. 247–262, 1975.

F.H. Clarke. *Optimization and Nonsmooth analysis*. Wiley, New York, 1983.

M.G. Cojocaru. *Projected Dynamical Systems on Hilbert Spaces*. PhD thesis, Departement of Mathematics and Statistics, Quenn's university, Kingston, Ontario, Canada, 2002.

M.G. Cojocaru & L.B. Jonker. Existence of solutions to projected differential equations on hilbert spaces. *Proceedings of the AMS*, 132 (1), pp. 183–193, 2003.

R. W. Cottle, J. Pang & R. E. Stone. *The linear complementarity problem*. Academic Press, Inc., Boston, MA, 1992.

K. Deimling. *Multivalued Differential Equations*. Walter de Gruyter, 1992.

Francisco Facchinei & Jong-Shi Pang. *Finite-dimensional variational inequalities and complementarity problems*, volume I & II of *Springer Series in Operations Research*. Springer Verlag NY. Inc., 2003.

A. F. Filippov. *Differential equations with discontinuous right hand sides*. Kluwer, Dordrecht, the Netherlands, 1988.

D. Goeleven & B. Brogliato. Stability and instability matrices for linear evolution variational inequalities. *IEEE Transactions on Automatic Control*, 49(4), pp. 521–534, 2004.

D. Goeleven, D. Motreanu, Y. Dumont & M. Rochdi. *Variational and Hemivariational Inequalities: Theory, Methods and Applications; Volume I: Unilateral Analysis and Unilateral Mechanics*. Nonconvex Optimization and its Applications. Kluwer Academic Publishers, 2003.

P.T. Harker & J.-S. Pang. Finite-dimensional variational inequality and complemntarity problems: a survey of theory, algorithms and applications. *Mathematical Programming*, 48, pp. 160–220, 1990.

W.P.M.H. Heemels. *Linear Complementarity Systems. A Study in Hybrid Dynamics*. PhD thesis, Technical University of Eindhoven, 1999. ISBN 90-386-1690-2.

W.P.M.H. Heemels & B. Brogliato. The complementarity class of hybrid dynamical systems. *European Journal of Control*, 9, pp. 311–349, 2003.

W.P.M.H. Heemels, J.M. Schumacher & S. Weiland. Linear complementarity problems. *S.I.A.M. Journal of applied mathematics*, 60 (4), pp. 1234–1269, 2000.

J.B. Hirriart-Urruty & C. Lemarechal. *Convex Analysis and Minimization Algorithms*, volume I et II. Springer Verlag, Berlin, 1993.

M. Johansson & A. Rantzer. Computation of piecewise quadratic lyapunov functions for hybrid systems. *IEEE Transactions on Automatic Control*, 43(4), pp. 555–559, 1998.

D. Kinderlehrer & G. Stampacchia. *An Introduction ot Variational Inequalities.* Academic Press, New York, 1980.

M. Kunze & M.D.P. Monteiro Marques. On parabolic quasi-variational inequalities ans state-dependent sweeping processes. *Topol. Methods Non Linear Analysis*, 12, pp. 179–191, 1998.

M. Kunze & M.D.P. Monteiro Marqus. An introduction to moreau's sweeping process. B. Brogliato, editor, *Impact in Mechanical systems: Analysis and Modelling*, volume 551 of *Lecture Notes in Physics*, pp. 1–60. Springer, 2000.

J.L. Lions & G. Stampacchia. Variational inequalities. *Communications on Pure ans applied Mathematics*, XX, pp. 493–519, 1967.

M. D. P. Monteiro Marques. *Differential Inclusions in NonSmooth Mechanical Problems : Shocks and Dry Friction*. Birkhauser, Verlag, 1993.

B.S. Mordukhovich. Generalized differential calculus for nonsmooth ans set-valued analysis. *Journal of Mathematical analysis and applications*, 183, pp. 250–288, 1994.

J.J. Moreau. Rafle par un convexe variable (première partie), exposé no 15. *Séminaire d'analyse convexe, University of Montpellier*, page 43 pages, 1971.

J.J. Moreau. Rafle par un convexe variable (deuxième partie) exposé no 3. *Séminaire d'analyse convexe, University of Montpellier*, page 36 pages, 1972.

J.J. Moreau. Evolution problem associated with a moving convex set in a Hilbert space. *Journal of Differential Equations*, 26, pp. 347–374, 1977.

J.J. Moreau. Liaisons unilatérales sans frottement et chocs inélastiques. *Comptes Rendus de l'Académie des Sciences*, 296 serie II, pp. 1473–1476, 1983.

J.J. Moreau. Unilateral contact and dry friction in finite freedom dynamics. J.J. Moreau & Panagiotopoulos P.D., editors, *Nonsmooth mechanics and applications*, number 302 in CISM, Courses and lectures, pp. 1–82. CISM 302, Spinger Verlag, 1988. Formulation mathematiques tire du livre Contacts mechanics.

D. Pang, J.-S. an Stewart. Differential variational inequalities. *Mathematical Programming A.*, 2006. submitted, preprint available at http://www.cis.upenn.edu/davinci/publications/pang-stewart03.pdf.

M. Schatzman. Sur une classe de problmes hyperboliques non linaires. *Comptes Rendus de l'Académie des Sciences Srie A*, 277, pp. 671–674, 1973.

M. Schatzman. A class of nonlinear differential equations of second order in time. *Nonlinear Analysis, Theory, Methods & Applications*, 2 (3), pp. 355–373, 1978.

G. Smirnov. *Introduction to the theory of Differential inclusions*, volume 41 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, 2002.

V.I. Utkin. Variable structure systems with sliding modes: A survey. *IEEE Transactions on Automatic Control*, 22, pp. 212–222, 1977.