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TRACKING CONTROL OF RIGID MANIPULATORS SUBJECT TO UNILATERAL CONSTRAINTS

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ABSTRACT

In this paper we study the tracking control of Lagrangian systems subject to frictionless unilateral constraints. More precisely it concerns a class of specific nonsmooth systems which perform cyclic impacting tasks. The stability analysis incorporates the hybrid and nonsmooth dynamical feature of the overall system. This work provides details on the conditions of existence of a such controller which guarantees an asymptotic strongly stability. Some tests in simulation give some result on the robustness of this controller. And finally it clarifies some concepts related to multiple impacting systems.

1 Introduction

The focus of this paper is the tracking control of a class of nonsmooth fully actuated Lagrangian systems subject to frictionless unilateral constraints on the position, $F(X) \geq 0$.

These nonsmooth complementarity systems evolve in three different phases :

- a free motion phase, where the mechanical system is not subject to any constraints (i.e. $F(X) > 0$),
- a permanently constraint phase where the dynamical system is subject to holonomic constraints ($F(X) = 0$),
- and a transition phase, where the manipulator is subject to multiple impacts and rebounds.

In the first phase the system is assumed to evolve in a free space and it is described by a set of ordinary differential equation. This problem has been solved since years by feedback linearization or by more specific controllers (adaptive control, robust control ...).

The second phase concerns permanently constrained system which are well known in force/position theory, see (McClamroch & Wang, 1988) for some example of force/position feedback algorithms.

During the transition phase the system is subject to unilateral constraints, and collisions occur. These collisions will generate rebounds, and rebounds are generally seen as disturbances in a feedback laws. In this control framework impacts are provoked intentionally to dissipate energy and contribute towards stabilizing the system.

The aim of this paper is to study a control scheme which guarantees some stability properties of the closed-loop system during general tasks. It is an extension of the framework developed in (Brogliato *et al.*, 2000) to the case of non-scalar frictionless unilateral constraint. We also study the robustness of this control scheme with respect to :

- the measurement noise on positions, velocities, accelerations,
- the knowledge of constraints position.

Finally we extend this work to the case of non scalar frictionless unilateral constraints.

1.1 Dynamics

The systems we study in this paper belong to the complementary-slackness class of hybrid dynamical systems (Van der Schaft & Schumacher, 2000) and are represented as:

$$
\begin{aligned}
M(X)\ddot{X} + C(X,\dot{X})\dot{X} + G(X) &= u + \nabla F(X).\lambda_X \\
F(X) \geq 0 \ , \ F(X).\lambda_X &= 0 \ , \ \lambda_X \geq 0 \\
&\textit{Collision rule}
\end{aligned}
\tag{1}
$$

Where $X \in \mathbb{R}^n$ is a vector of generalized coordinates, $M(X) \in \mathbb{R}^{n \times n}$ is the inertia matrix, $F(X) \in \mathbb{R}^m$ represent the distance to the constraints, $\lambda_X \in \mathbb{R}^m$ are the Lagrangian multipliers associate to each constraints, $u \in \mathbb{R}^{n_u}$ is the vector of generalized torques. $C(X,\dot{X})$ is the matrix of Coriolis and centripedal forces, $G(X)$ contains conservative forces.

1.2 Impact model

A collision rule is needed to make the system (1) integrable. A collision rule is a relation between the post impact velocities and the pre-impact velocities. In all the paper impact times are denoted as $t_k,\ k \geq 0$.

In this work, the collision rule used is chosen as (Moreau, 1988) :

$$
\begin{gathered}
\dot{X}(t_k^+) = -e_n \dot{X}(t_k^-) + \\
(1+e_n) \underset{z \in V(X(t_k))}{\operatorname{argmax}} \frac{1}{2}(z - \dot{X}(t_k^-))^T M(X(t_k))((z - \dot{X}(t_k^-))
\end{gathered}
\tag{2}
$$

where $\dot{X}(t_k^+)$ is the post impact velocities, $\dot{X}(t_k^-)$ is the pre-impact velocities, $V(X(t_k))$ the tangent cone (see figures 1,2) and e_n is the restitution coefficient of the surface, $e_n \in [0,1]$.

When $m = 1$, (2) is the Newton's Law ($X_n(t^+) = -e_n X_n(t^-)$).

1.3 Admissible space

The admissible domain Φ is a closed domain in the configuration space where the system can involve;

$$
\Phi = \{X, F(X) \geq 0\} = \bigcap_i \Phi_i,\ \Phi_i = \{X, F_i(X) \geq 0\}
$$

Definition 1. *A singularity of $\partial\Phi$ is the intersection of two (or more) surfaces $\Sigma_i = \{X, F_i(X) = 0\}$.*

The stabilization of the dynamic system is studied only onto a singular convex point (like on figure 1.a). Unilateral constraint is expressed by the relation $F(X) \geq O$, which can be translated locally by the system : $\begin{cases} C_1 X_1 \geq 0 \\ \vdots \\ C_m X_m \geq 0 \end{cases}$. Clearly the non-convex

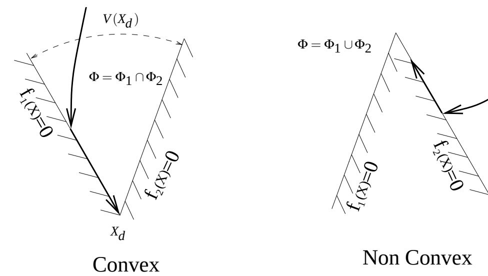

Figure 1. Singular points

exemple of figure 1.(b) cannot be expressed as the intersection of convex domains Φ_i. This case in named a reintrant corner in the literature, and modelling issues are not yet fixed for reintrant corners (Glocker, 2001) (Fremond, 2002)

This restriction on singular non-convex points don not mean that the whole space must be convex, for exemple the domain of the figure 2 is non convex but the stabilization of a dynamic system can be study on every point of $\partial\Phi$

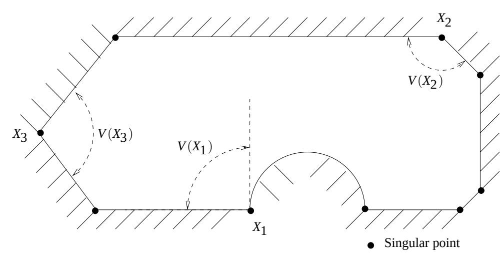

Figure 2. Exemple of a suitable non convex domain

1.4 Cyclic impacting robotic task

In this paper we restrict ourselves to a specific task : a succession of free and constrained phases Ω_k.

During the transition between a free and a constrained phase, the dynamic system passes into a transition phase I_k. In the phase I_k, the system is subject to collisions. Transition between constrained and free motion is smooth.

A robotic task with cyclic impact phases can be represented in the time domain as :

$$
\mathbb{R}^+ = \Omega_0 \cup I_0 \cup \Omega_1 \cup \Omega_2 \cup I_1 \cup \ldots \cup \Omega_{2k-1} \cup \Omega_{2k} \cup I_k \cup \ldots \tag{3}
$$

where Ω_{2k} denotes the free-motion phases and Ω_{2k+1} denotes constrained-motion phases.

2 Stability framework

The systems studied in this paper are complex hybrid dynamical system which involve continuous as well as discrete time phases. The stability criterion used in this paper is an extension of the Lyapunov second method for closed loop mechanical system with unilateral constraints proposed in (Brogliato *et al.*, 1997) and (Brogliato *et al.*, 2000)

The main idea of this criterion is to get simultaneously continuous and discrete stability via the same Lyapunov's function V. The criterion guarantees that the collisions during impact phases do not destroy the Lyapunov stability of the closed loop system.

Definition 2 (Ω-weakly stable system). *The closed-loop system is Ω-weakly stable if for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\| x(0) \| \leq \delta(\varepsilon) \Rightarrow \| x(t) \| \leq \varepsilon$ for all $t \geq 0$, $t \in \Omega$.*

Let us define the closed-loop impact Poincaré maps that correspond to the section $\Sigma_i^+ = \{x : F_i(X) = 0, \dot{X}^T \nabla_X F_i > 0\}$:

$$\begin{aligned} P_\Sigma : \Sigma_i^+ &\rightarrow \Sigma_i^+, \\ x_{\Sigma,i}(k) &\mapsto x_{\Sigma,i}(k+1). \end{aligned} \tag{4}$$

Definition 3 (Strongly stable system). *The system is said strongly stable if: (i) it is Ω-weakly stable, (ii) P_Σ is Lyapunov stable, and (iii) the sequence $\{t_k\}_{k\in\mathbb{N}}$ has a finite accumulation point $t_\infty < +\infty$.*

Let us define the jump function $\sigma_f(t) = f(t^+) - f(t^-)$ and $\lambda[.]$ is the Lebesgue measure.

Claim 1 (Weak Stability). *Assume that*

- *the task is as in (3),*
- $\lambda[\Omega] = +\infty$,
- *for each $k \in \mathbb{N}$, $\lambda[I_k] < +\infty$,*
- $V(x(t_f^k), t_f^k) \leq V(x(t_0^k), t_0^k)$,
- *$V(x(.), .)$ uniformly bounded on each I_k.*

If on Ω, $\dot{V}(x(t),t) \leq 0$ and $\sigma_V(t_k) \leq 0$ for all $k \geq 0$, then the closed-loop system is Ω-weakly stable. If $\dot{V}(x(t),t) \leq -\gamma(\| X \|)$ for some class K function γ, then the system is asymptotically Ω-weakly stable.

Claim 2 (Strong Stability). *The system is strongly stable if:*

- *it is weakly stable,*
- $\sigma_V(t_k) \leq 0$
- $V(t_{k+1}^-) \leq V(t_k^+)$;
- *V is uniformly bounded and time continuous on $I_k - \cup_k\{t_k\}$, where the sequence $\{t_k\}$ exists and has a finite accumulation point.*

As we will see, getting the asymptotic strong stability for (1) subject to (3), is a hard task in general.

3 Tracking controller framework

To make the controller design easier the dynamical equations (1) are considered in the generalized coordinates introduced in (McClamroch & Wang, 1988). After transformation in the new coordinates, the dynamic system is as follows :

$$q = \begin{bmatrix} q1 \\ q2 \end{bmatrix} \;,\; q1 = \begin{bmatrix} q_1^1 \\ \vdots \\ q_1^m \end{bmatrix} \;,\; q = Q(X)$$

$$\begin{aligned} M_{11}(q)\ddot{q_1} + M_{12}(q)\ddot{q_2} + C_1(q,\dot{q})\dot{q} + g1(q) &= T_1(q)u + \begin{pmatrix} \lambda_{q_1^1} \\ \lambda_{q_1^2} \end{pmatrix} \\ M_{21}(q)\ddot{q_1} + M_{22}(q)\ddot{q_2} + C_2(q,\dot{q})\dot{q} + g2(q) &= T_2(q)u \\ q_1^1 \geq 0 \;,\; q_1^1.\lambda_{q_1^1} = 0 \;,\; \lambda_{q_1^1} &\geq 0 \\ &\cdots \\ q_1^m \geq 0 \;,\; q_1^m.\lambda_{q_1^m} = 0 \;,\; \lambda_{q_1^m} &\geq 0 \end{aligned} \tag{5}$$

The controller developed in this paper uses three different low-level control laws for each phase (Ω_{2k}, Ω_{2k+1} and I_k) :

$$\begin{aligned} u &= U_{nc} \text{ for } t \in \Omega_{2k} \\ u &= U_t \text{ for } t \in I_k \\ u &= U_c \text{ for } t \in \Omega_{2k+1} \end{aligned} \tag{6}$$

A supervisor switches between this three control laws, and the supervisor must generate desired tracking trajectories to ensure the stability. The objective of the controller is to stabilize the system on the codimension m surface $\cap_i \Phi_i$ during transition phases I_k, and to assure the stability in the sense of definition 2 and 3 above.

The stability of this controller is analyzed by using the criteria proposed in section 2. The key of this analysis is to prove that the Lyapunov function has decreased between two consecutive rebounds.

During Ω_k phase (free or permanently constraint motion) the tracking reference trajectories are determined in regard to the task we want to accomplish, whereas during I_k desired reference trajectories are determined to ensure the stabilization on the constrained surface.

The asymptotic stability of this scheme makes the system land on the constraint surfaces tangentially after enough cycles of constraints/free motions (one cycle = $\Omega_{2k} \cup I_k \cup \Omega_{2k+1}$). Asymptotically the transitions between free motion phase and permanently constraint phase are done without any rebound.

For this control framework, we use a Lyapunov function which is very close to the nonsmooth global energy of the system. Let us notice that the usefulness of using Poincaré mapping during transition phases, partially lies in the fact that even in some simple cases guaranteeing $\dot{V}(t) \leq 0$ for all t is not possible, see (Brogliato *et al.*, 2000).

$$V(t,\tilde{q},\dot{\tilde{q}}) = \frac{1}{2}\dot{\tilde{q}}^T M(q)\dot{\tilde{q}} + \frac{1}{2}\gamma_1\tilde{q}^T\tilde{q} + \Psi_\Phi(q) \qquad (7)$$

with $\tilde{q} = q - q_d$ where q_d is a desired trajectory, and $\Psi_\Phi(q)$ is the indicator function of the set $\Phi = \{q : F(q) \geq 0\}$ (Moreau, 1988); $\Psi_\Phi(q) = \begin{cases} 0 & if\ q \in \Phi \\ +\infty & if\ q \notin \Phi \end{cases}$ which is a nonsmooth potential associated to the unilateral constraints.

The control law used in this scheme is an extension of the asymptotically stable controller of (Paden and Panja, 1988), originally designed for free-motion position and velocity tracking. Let us propose (see (6))

$$\begin{aligned} U_{nc} &= M(q)\ddot{q}_d + C(q,\dot{q})\dot{q}_d + g(q) - \gamma_1\tilde{q} - \gamma_2\dot{\tilde{q}} \\ U_t &= U_{nc} \quad \text{before the first impact} \\ U_t &= g(q) - \gamma_1\bar{q} - \gamma_2\dot{q} \quad \text{after the first impact} \\ U_c &= U_{nc} - P_d + K_f(P_q - P_d) \\ & \gamma_1 > 0 \ \ \gamma_2 > 0 \ \ K_f > 0 \end{aligned} \qquad (8)$$

where P_d is the desired force we want for the permanently constraint motion. The interest for choosing this controller is that the function $V(t,\tilde{q},\dot{\tilde{q}})$ in (7) is very close to the total energy of the system.

A first observation is that a control strategy which consists of attaining the surface $\partial\Phi$ tangentially and without incorporating impacts in the stability analysis, cannot work in pratice due to its lack of robustness (because it implies the perfect knowledge of the constraint location).

In view of this, the control law for the transition phase is defined in order :

- to make the system hit the constraint surface (and then dissipate energy during impacts) if the tracking error is too large. If $V(t,\tilde{q},\dot{\tilde{q}})$ is large, q_d is computed to hit the constraint surface.
- To make the system approach the constraint surface tangentially (without rebound) if the tracking is perfect. If $V(t,\tilde{q},\dot{\tilde{q}})$ is small, q_d is computed to land smoothly on the surface.

This two situations are conflicting because we want impacts for the robustness of the control: indeed with each collision there is an kinetic energy loss which stabilizes the system provided the controller is suitably chosen. On the other hand the coupling between q_1 and q_2 in (5), and the stability framework in claims 1 and 2, make the asymptotic stability quite difficult to obtain if velocities are subject to discontinuities. This conflicting situation implies a specific controller to achieve the required stability properties.

During the transition phase $q_d(t)$ is defined as follows (see figure 3) :

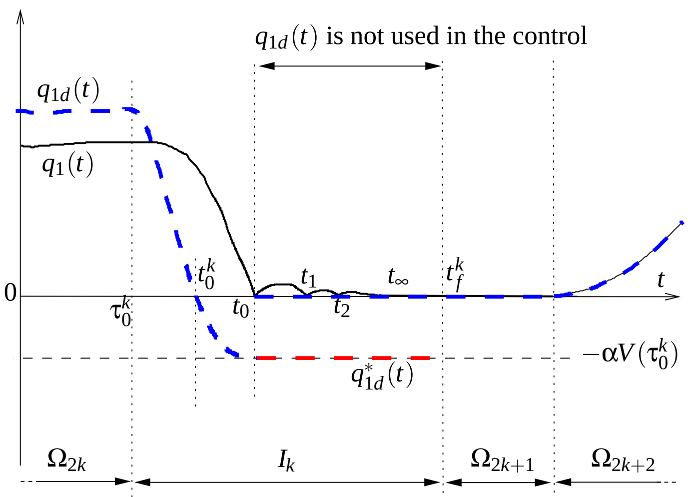

Figure 3. Desired trajectory q_{1d}

Let us define :

- τ_0^k is the chosen by the designer as the start of the transition phase,
- t_0^k is the time corresponding to $q_{1d}(t_0^k)$=0,
- t_0 corresponds to the first impact,
- t_∞ correspond to the finite accumulation point.
- t_f^k is the end of the transition phase

On $[\tau_0^k, t_0)$, we impose that $q_d(t) \in C^2$ and $q_{1d}(t)$ decreases towards $-\alpha V(\tau_0^k)$.

On $[t_0, t_f]$, we define q_d and q_d^* as follows :

$$q_d = \begin{pmatrix} 0 \\ q_{2d}^* \end{pmatrix}, \ q_d^* = \begin{pmatrix} -\alpha V(\tau_o^k) \\ q_{2d}^* \end{pmatrix}$$

where q_d is the fixed point of the whole system (the Lagrangian equation and the unilateral constraint), and q_d^* is the fixed point of the closed loop system without the constraint. Indeed q_d^* is outside the admissible domain Φ, and cannot be reached by the system. The purpose of q_d^* is to create a "virtual" potential force which stabilizes the system on $\partial\Phi$ even if the position of the constraint is uncertain.

Consequently the real fixed point of the system $(q_d, \dot{q}_d)$ is used in the expression of the Lyapunov fonction : $\tilde{q} = q - q_d$, whereas the unreachable fixed point q_d^* is used in the control law : $\bar{q} = q - q_d^*$.

In order to cope with coupling between q_1 and q_2, the signal $q_{2d}(t)$ is frozen during the transition phase, i.e.:

- $q_{2d}(t) = q_{2d}^*$, $\dot{q}_{2d}(t) = 0$ on $[t_0^k, t_f]$
- $q_{2d}(t)$ is defined such that $\dot{q}_{2d}(t_0^{k-}) = 0$ on $[\tau_0^k, t_0^k]$

Let τ_1^k be such that $q_{1d}(\tau_1^k) = -\alpha V(\tau_0^k)$ and $\dot{q_{1d}}(\tau_1^k) = 0$.

Definition 4. *$\{CI\}$ is the subspace of initial condition $X(0)$ which assure $t_0 \geq \tau_1^k$.*

Let $m = 1$ in (1).

Claim 3. *The system defined by (1) in closed-loop with the controller in (8) and $q_d(t)$ as defined above, is :*

- *Asymptotically strongly stable if $X(0) \in \{CI\}$ and if $\gamma_1 > 0$, $\gamma_2 > 0$*
- *Asymptotically weakly stable if $X(0) \notin \{CI\}$ and if γ_1, γ_2 and α are large enough.*

Proof. The proof of the first item can be found in (Brogliato *et al.*, 2000). The proof of the second item follows the same line but in this case $\sigma_V(t_0)$ is not negative. One has :

$$\sigma_V(t_0) = T_L(t_0) - \frac{1}{2}\gamma_1 q_{1d}{}^2(t_0^-) - \frac{1}{2}\dot{q_d}(t_0^-)^T M\dot{q_d}(t_0^-) + \dot{q}(t_0^-)^T M\dot{q_d}(t_0^-)$$

where $T_L(t_0)$ is the loss of kinetic energy at the impact t_k.
To obtain the weak stability, it is sufficient that :

$$V(t_1^-) - V(t_0^-) \leq 0 \tag{9}$$

Inequality (9) is equivalent to :

$$\sigma_V(t_0) - \gamma_2 \int_{t_0^+}^{t_1^-} \dot{q}^T \dot{q} dt \leq 0 \tag{10}$$

And (10) is implied by :

$$\dot{q}(t_0^-)^T M\dot{q_d}(t_0^-) \leq \gamma_2 \int_{t_0^+}^{t_1^-} \dot{q}^T \dot{q} dt \tag{11}$$

Or, integrating the right-hand side of (11) by parts, by :

$$\dot{q}(t_0^-)^T M\dot{q_d}(t_0^-) \leq -\gamma_2 \int_{t_0^+}^{t_1^-} q^T \ddot{q} dt \tag{12}$$

Let us calculate an estimation of the integral in (12). By linearization around the point $q = \begin{pmatrix} 0 \\ q_{2d}^* \end{pmatrix}$, the closed-loop equation of the system is :

$$M\ddot{q} + \gamma_1 \begin{pmatrix} \alpha V(\tau_o^k) \\ 0 \end{pmatrix} + \gamma_2 \dot{q} = 0 \tag{13}$$

Inserting (13) in (12) gives the inequality :

$$\dot{q}(t_0^-)^T M\dot{q_d}(t_0^-) \leq \gamma_1\gamma_2 \int_{t_0^+}^{t_1^-} q^T M^{-1} \begin{pmatrix} \alpha V(\tau_o^k) \\ 0 \end{pmatrix} dt + \frac{\gamma_2^2}{2}[q^T M^{-1} q]_{t_0}^{t_1} \tag{14}$$

$$[q^T M^{-1} q]_{t_0}^{t_1} = 0 \text{ because } q(t_k) = 0 \tag{15}$$

Therefore (14)implies :

$$\dot{q}(t_0^-)^T M\dot{q_d}(t_0^-) \leq \gamma_1\gamma_2\alpha V(\tau_o^k) \int_{t_0^+}^{t_1^-} (M_{21}^{-1})^{\,T} q_2 dt \tag{16}$$

If the coupling between q_1 and q_2 is weak, we have $q_2(t) = q_{2d}^*$ on $[t_0^+; t_1^-]$, and (16) becomes :

$$\dot{q}(t_0^-)^T M\dot{q_d}(t_0^-) \leq (t_1 - t_0)\gamma_1\gamma_2\alpha V(\tau_o^k)(M_{21}^{-1})^{\,T} q_{2d}^* \tag{17}$$

Let Δ_{min} be a lower bound of $\Delta = (t_1 - t_0)$. Then (17) implies :

$$\dot{q}(t_0^-)^T M\dot{q_d}(t_0^-) \leq \gamma_1\gamma_2\alpha V(\tau_o^k)(M_{21}^{-1})^{\,T} q_{2d}^* \Delta_{min} \tag{18}$$

To prove that it exist γ_1, γ_2 and α which verify (18), we need to verify that $\dot{q}(t_0^-)^T M\dot{q_d}(t_0^-)$ is bounded :

On $[\tau_0^k; t_0]$, $U_t = U_{nc}$ and $\dot{V}(t) \leq 0$, then we have :

$$\begin{aligned}
V(\tau_0^k) &\geq V(t_0^-) \\
V(\tau_0^k) &\geq \frac{1}{2}\dot{\tilde{q}}(t_0^-)^T M(q)\dot{\tilde{q}}(t_0^-) + \frac{1}{2}\gamma_1 \tilde{q}(t_0)^T \tilde{q}(t_0) \\
V(\tau_0^k) &\geq \frac{1}{2}\dot{\tilde{q}}(t_0^-)^T M(q)\dot{\tilde{q}}(t_0^-) \\
V(\tau_0^k) &\geq \frac{1}{2}\lambda_{min}[M(q(t))] \parallel \dot{q}(t_0^-) - \dot{q}_d(t_0^-) \parallel^2
\end{aligned} \tag{19}$$

From (19) we have :

$$\parallel \dot{q}(t_0^-) \parallel \leq \left(\frac{2V(\tau_0^k)}{\lambda_{min}[M(q(t))]}\right)^{1/2} + \max_{t\in[\tau_0^k; t_0]} \parallel \dot{q}_d(t) \parallel \tag{20}$$

Hence $\dot{q}(t_0^-)$ is bounded, $\dot{q}_d(t_0^-)$ is bounded by $\max_{t\in[\tau_0^k;t_0]} \dot{q}_d(t)$. Then $\dot{q}(t_0^-)^T M\dot{q}_d(t_0^-)$ is bounded. If $\Delta_{min} \neq 0$, there exist γ_1, γ_2 and α such that (18) is satisfied.

To calculate Δ_{min}, we need to solve equation (13). The solution is of the form :

$$q_1(t) = A.e^{-\frac{\gamma_2}{M_{11}}t} + B.t + C$$
$$\text{with } A = -\frac{M_{11}}{\gamma_2}\left(\frac{\gamma_1}{\gamma_2}\alpha V(\tau_o^k) + \dot{q}_1(t_0^+)\right) \ ,$$
$$B = -\frac{\gamma_1}{\gamma_2}\alpha V(\tau_o^k) \ , \ C = \frac{M_{11}}{\gamma_2}\left(\frac{\gamma_1}{\gamma_2}\alpha V(\tau_o^k) + \dot{q}_1(t_0^+)\right)$$

$\Delta \geq 0$ is a solution of $q_1(t) = 0$. A developpement of order 2 gives a lower bound.

$$\Delta_{min} = \frac{2M_{11}\dot{q}_1(t_0^+)}{\gamma_1\alpha V(\tau_o^k) + \gamma_2\dot{q}_1(t_0^+)} \tag{21}$$

$\star$ If $1 > e_n > e_\varepsilon$, then $\dot{q}_1(t_0^+) = -e_n\dot{q}_1(t_0^-) > 0$ and $\Delta_{min} > 0$, we can conclude that there exist α, γ_1 and γ_2 large enough such that inequality (9) holds.

$\star$ If $0 \leq e_n < e_\varepsilon$, then $\dot{q}_1(t_0^+)$ and Δ_{min} are very small. One has :

$$\lim_{e_n \to 0}(t_\infty - t_0) = 0 \tag{22}$$

and the sufficient condition (18) gives that :

$$\lim_{e_n \to 0}(\alpha\gamma_1\gamma_2) = +\infty \tag{23}$$

In this case we cannot conclude on the existence of bounded α, γ_1 and γ_2. However it exist a bounded $\delta > 0$, such as for $t_f^k > t_\infty + \delta$ we have :

$$V(t_f^k) - V(t_0^-) \leq 0 \tag{24}$$

Inequality (24) assures the weak-stability in this last case.

4 Simulation & Robustness study

This control scheme is tested in simulation on a 2-link planar manipulator for the simplest case of a scalar constraint, with a Newton's restitution rule. The constraint surface corresponds to the ground ($y = 0$). The natural generalized coordinates so that the dynamics fits with (5), with $m = 1$, are the work-space coordinates (x, y). We take:

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \ , \ y > 0$$

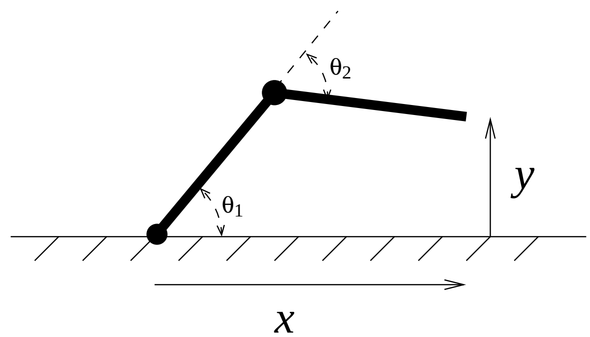

Figure 4. 2-link planar manipulator

4.1 Evolution of V(t) at impact times

4.1.1 First impact after that q_d was totally frozen

In this simulation $X(0) \in \{CI\}$, and the impact occurs when $q_{1d} = 0$. This means that at the first impact time t_0, the variation of the Lyapunov function σ_V is negative.

$$\sigma_V(t_0) = T_L(t_0) - \tfrac{1}{2}\gamma_1 {q_{1d}}^2(t_0^-) < 0$$

On figure 5, $V(t)$ decreases and σ_V is negative, this means that the claim 3 applies and the closed loop system is asymptotically strongly stable.

4.1.2 First impact before that q_d was totally frozen

In this simulation $X(0) \notin \{CI\}$, and the impact occurs when $q_{1d} \neq 0$. This means that at the first impact time t_0, the variation of the Lyapunov function σ_V can be positive in function of $\dot{q}(t_0^-)^T M\dot{q}_d(t_0^-)$.

$$\sigma_V = T_L(t_0) - \frac{1}{2}\gamma_1 {q_{1d}}^2(t_0^-) - \frac{1}{2}\dot{q}_d(t_0^-)^T M\dot{q}_d(t_0^-) + \dot{q}(t_0^-)^T M\dot{q}_d(t_0^-)$$

On figure 6, $V(t)$ is not decreasing all the time: at the first impact time of each cycle the Lyapunov function has a peak. This

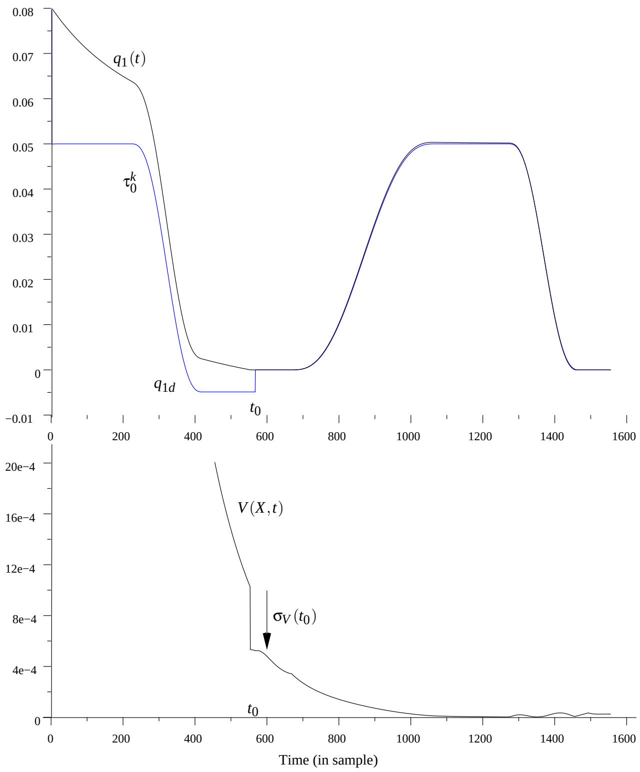

Figure 5. Negative jump of $V(t)$

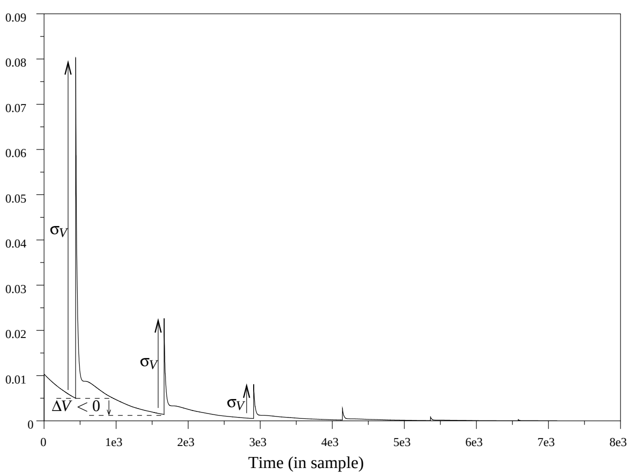

Figure 6. Positive jump of $V(t)$

means that the strong stability of the closed-loop system is not assured, but if γ_1, γ_2 and α are chosen like in claim 3, the system is weak-stable. We see that $V(\tau_{n+1}) - V(\tau_n) < 0$ between two consecutives cycles.

4.2 Asymptotic convergence

Figure 7 shows the evolution of $q_1(t)$ and $q_2(t)$ during cyclic tasks. On the graph of q_1, the asymptotic convergence of the controller is exhibited as the value of $\alpha V(q,t)$ decreases exponentially.

The graph of q_2 shows the coupling between q_1 and q_2. At each impact time a jump in $\dot{q}_2$ occurs. The periodic step on q_{2d} corresponds to the transition phase during which q_{2d} needs to be frozen.

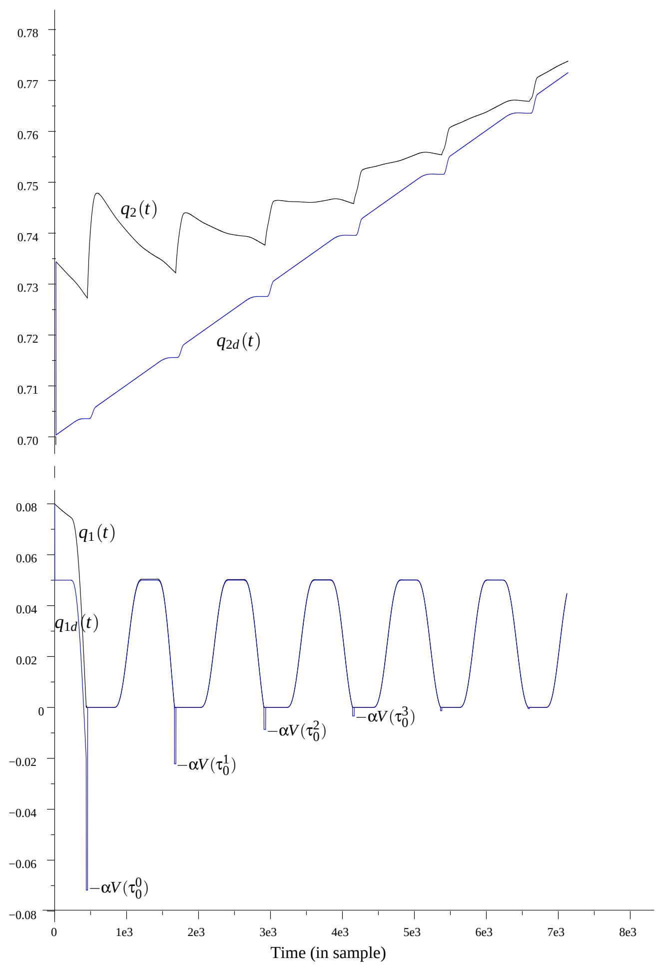

Figure 7. Asymptotic Convergence

Figure 8 illustrates the asymptotic stability of this control strategy. The simulation begins with an important error position tracking error. The first impacts have a large magnitude (function of $V(\tau_k^0)$) and after only five cycles rebounds practically vanish. Asymptotically the transition phase disappears.

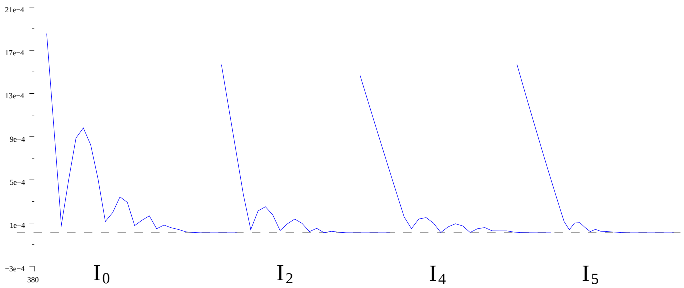

Figure 8. Attenuation of rebounds height.

4.3 Robustness

In this subsection, we study the robustness of the controller with respect to the uncertainty on the constraint position and to the noise on the measured state vector.

4.3.1 uncertainty on constraint position

In this subsection the position of the constraint surface is not known precisely. As seen on figure 9 there are two situations to consider.

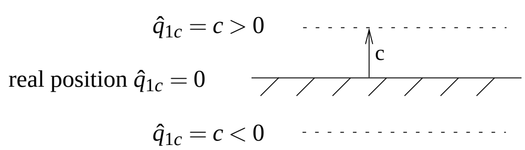

Figure 9. estimated position $\hat{q}_{1c}$

$\star$ If $c < 0$, the estimated position of the constraint is lower than the real position. In this case the desired trajectories decrease toward $q_{1d} = -\alpha V(\tau_k^0) - |c|$ instead of $q_{1d} = -\alpha V(\tau_k^0)$. The error c can be incorporated in the term $-\alpha V(\tau_k^0)$ and the stability of the transition phase is not changed. During the constraint phase the controller is :

$$U_c = U_{nc}{}^{ideal} - (P_d + \gamma_1 \begin{bmatrix} |c| \\ 0 \end{bmatrix}) + K_f(P_q - P_d)$$

The error term $\gamma_1 |c|$ is added to the desired force P_d and contributes to keep the contact with the surface during the constrained phase.

In this case, the stability of the system is not changed, but the system loses is asymptotic stability : If the tracking is perfect $V(\tau_k^0) = 0$ and $q_{1d}^* = -|c|$, then the system does not approach the surface tangentially and rebounds occur. The asymptotic stability is not preserved. Figure 10 shows an exemple of

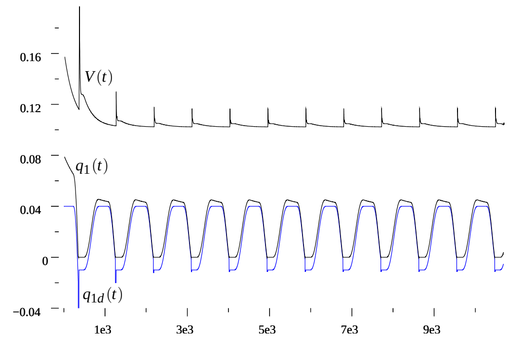

Figure 10. Stability if $c < 0$

stabilization where $c < 0$.

$\star$ If $c > 0$, the estimated position of the constraint is above the real position. If the tracking is perfect $V(\tau_k^0) = 0$, the desired trajectory decreases toward $q_{1d} = c$ and the system never reaches the constraint. There is no convergence (see figure 11).

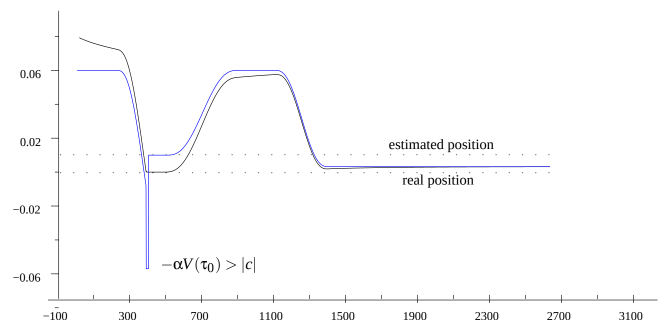

Figure 11. Non convergence if $c > 0$

This problem can be solved by monitoring the time of stabilization. If there is no stabilization after the time $\hat{t}_\infty$, the estimated position of the constraint is refresh $\hat{q}_{1c}^{new} = \hat{q}_{1c}^{old} - \varepsilon$. After a finite number of iterations, one gets $\hat{q}_{1c} < 0$. The system is in the previous situation $c < 0$ and the stability is preserved. Figure 12 show an example of auto adjustment of the estimated position of the constraint.

When the tracking is not perfect and $\alpha V(\tau_k^0) > c$, the previous problem is not present, the transition phase is able to stabilize the system on the surface. But during the constraint phase, the control law is:

$$U_c = U_{nc}{}^{ideal} - (P_d - \gamma_1 \begin{bmatrix} c \\ 0 \end{bmatrix}) + K_f(P_q - P_d)$$

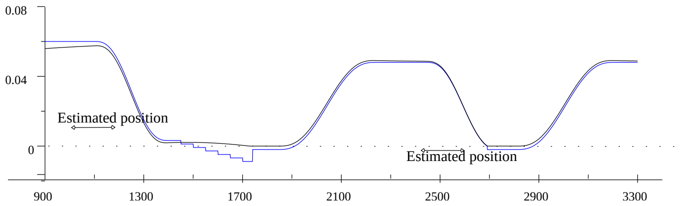

Figure 12. Auto adjustment of $\hat{q}_{1c}$

P_d must be chosen enough large in regard to $\gamma_1 c$ to be sure that system keeps the contact with the surface during all the constraint phase.

4.3.2 noise measurement on state vector

We consider noise on the measurement of the position :

$$\hat{q} = q + b$$

where b is the noise, assumed to be a white noise.

The control law U_t becomes :

$$U_t = g(q) - \gamma_1(\hat{q} - q_d^*) - \gamma_2 \dot{q}$$
$$U_t = g(q) - \gamma_1 \begin{pmatrix} q_1 - (-\alpha V(\tau_0^k) - b) \\ q_2 + b - q_{2d}^* \end{pmatrix} - \gamma_2 \dot{q}$$

q_{1d}^* needs to be negative to attain the surface. Then we need $|b| < \alpha V(\tau_0^k)$ to have convergence. To avoid this problemn we need to have α enough large.

Figure 13 shows a simulation where α is too small: After three cycles the value of $\alpha V(\tau_0^2)$ no longer dominates the noise. The switching time t_f^k between I_k and Ω_{2k+1} is defined to be as $t_k^k \geq t_\infty$. Therefore in this case there is no stabilization on the surface and $t_\infty = +\infty$. The cycling tasks in (3) are stopped.

Figure 14 shows a simulation where α is larger, in this situation impacts always occur.

5 Mutiple impact

This section extend the previous controller framework to the case of multiple impact.

Definition 5 (Multiple impact). *A multiple impact is an impact into a singularity as in definition 1. If the singularity has codimension α, the multiple impact is named an α-impact.*

At multiple impacts, more than one relation of the form $F_i(q) = 0$ are verified at the same time.

The difficulty created by stabilization at singularities of $\partial\Phi$, is that the way the system attains the singularity, may vary: the

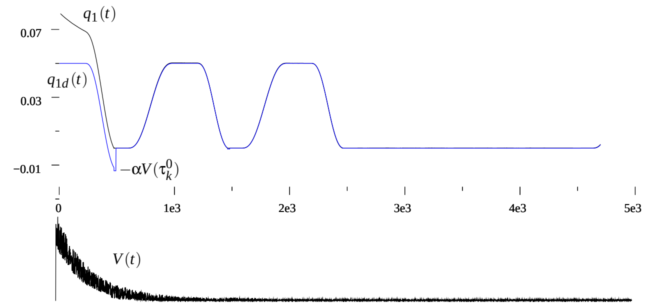

Figure 13. α too small with respect to the noise level

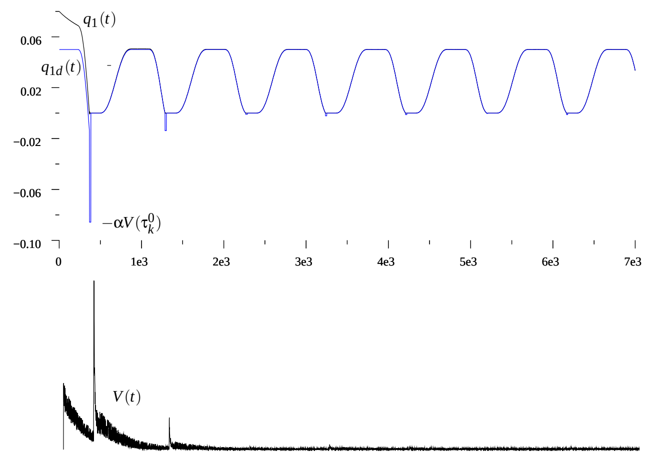

Figure 14. Suitable α

system may hit the singularity directly, or hit one or several surfaces Σ_i (through a finite or infinite number of impacts) before attaining the singularity.

5.1 Stability of transition controller if $\theta_{kinetic} < \frac{\pi}{2}$

In this subsection, we want to stabilize the system on the sub-space $\Sigma_1 \cap \Sigma_2$. In a first instance we restrict ourselves to an admissible space Φ where the angle between the surfaces is less than $\frac{\pi}{2}$ in the kinetic metric. Figure 15 depicts the situation in the planar case, or can be viewed as an abstract section of the configuration space.

The kinetic angle is the angle in the kinetic metric defined as $x^T M(q) y$ for vectors x and y. The reason why we make a difference between $\theta_{kinetic} > \frac{\pi}{2}$ and $\theta_{kinetic} \leq \frac{\pi}{2}$, is that (at least in the planar case and a 2-impact) $\theta_{kinetic} > \frac{\pi}{2}$ implies discontinuity of the solutions with respect to initial data, whereas with $\theta_{kinetic} \leq \frac{\pi}{2}$ solutions are continous in the initial data $(q(0), \dot{q}(0))$. This is expected to influence the stabilization strategy on the singularity.

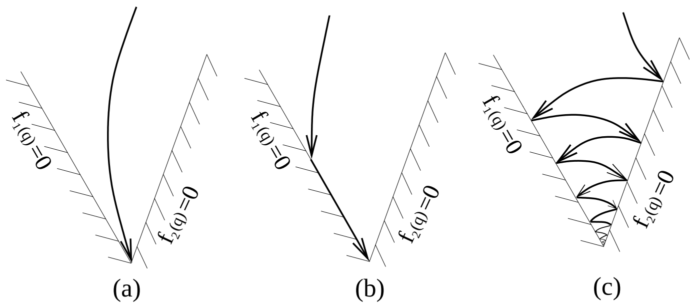

Figure 15. Multiple impacts (2-impacts)

5.1.1 Case (a) : Simultaneous impacts

In this case, the multiple impact on the surfaces are simultaneous, this means that at each impact time t_k, $q_1^1(t_k) = q_1^2(t_k) = 0$, and the closed loop analysis made in (Brogliato *et al.*, 2000) for a 1-impact can be adapted immediately to α-impact, $\alpha \geq 2$. Indeed at each impact time t_k, $q_1(t_k) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

5.1.2 Case (b) : Impacts on one surface before double impact

In this case the stabilization is made in two steps. A first step where the system is stabilized on the first surface $f_1(q) = 0$ (without impact on the second surface). And a second phase during which the constraint system on $f_1(q) = 0$ is stabilized on both surfaces.

The proof of stability of the fisrt phase is the same as for the 1-impact case if we take $q_1 = [q_1^1]$ and $q_2 = \begin{bmatrix} q_1^2 \\ q_2 \end{bmatrix}$.

During the second phase, the system is in a constraint motion, and its closed loop dynamic equation is:

$$M(q)\ddot{q} = -C(q,\dot{q})\dot{q} - \gamma_1\bar{q} - \gamma_2\dot{q} + (1+k_{f1})(\lambda_{q_1} - \lambda_q)\nabla_q q_1^1 \quad (25)$$

Then the system is stabilized on both surfaces using desired trajectories $q_{1d} = \begin{bmatrix} 0 \\ q_{1d}^{2*} \end{bmatrix}$, where q_{1d}^{2*} has the same form as q_{1d}^{1*} in the previous phase and decreases toward $-\alpha_2 V(X(\tau_0^k), \tau_0^k)$.

With the same demonstration as before, we need to prove that :

$$V(X(t_{k+1}^-), t_{k+1}^-) - V(X(t_k^+), t_k^+) \leq 0 \quad (26)$$

One obtains :

$$
\begin{aligned}
& V(X(t_{k+1}^-), t_{k+1}^-) - V(X(t_k^+), t_k^+) \\
&= \int_{t_k^+}^{t_{k+1}^-} \dot{V}(t)dt \\
&= \int_{t_k^+}^{t_{k+1}^-} \dot{q}^T M\ddot{q} + \dot{q}^T \frac{\dot{M}}{2}\dot{q} + \gamma_1\tilde{q}^T\dot{\tilde{q}}dt \\
&= \int_{t_k^+}^{t_{k+1}^-} \{\dot{q}^T[-C\dot{q} - \gamma_1\bar{q} - \gamma_2\dot{q} + (1+k_{f1})(\lambda_{q_1} - \lambda_q)\nabla_q q_1^1] \\
&\qquad + \dot{q}^T \frac{\dot{M}}{2}\dot{q} + \gamma_1\tilde{q}^T\dot{\tilde{q}}\}dt \quad (27) \\
&= \int_{t_k^+}^{t_{k+1}^-} -\gamma_2\dot{q}^T\dot{q}dt + \gamma_1\int_{t_k^+}^{t_{k+1}^-} \dot{q}_1^T q_{1d}^* dt \\
&\qquad + \int_{t_k^+}^{t_{k+1}^-} \dot{q}^T(1+k_{f1})(\lambda_{q_1} - \lambda_q)\nabla_q q_1^1 dt \quad (28) \\
&= \int_{t_k^+}^{t_{k+1}^-} -\gamma_2\dot{q}^T\dot{q}dt \leq 0 \quad (29)
\end{aligned}
$$

(28) is deduced from (27) since $2C - \dot{M}$ is skew-symmetric and $\dot{\tilde{q}}^T\tilde{q} - \dot{q}^T\bar{q} = \dot{q}^T q_{1d}^*$. (29) is deduced from (28) since $\dot{q}^T(1 + k_{f1})(\lambda_{q_1} - \lambda_q)\nabla_q q_1^1 = 0$ and $[q_1^T q_{1d}^*]_{t_k}^{t_{k+1}} = 0$ since $q_1(t_k) = 0$ during the 2-impact.

Equality (29), and a demonstration like in the 1-impact case give the asymptotic stability of this 2-impact. But in this case we have supposed that there is no impact on the second surface before the 2-impact.

5.1.3 Case (c) : General case

In this case the system can collide indifferently the two surfaces. There are several 1-impacts on the both surfaces before the 2-impact occurs. In this situation we do not have $q_1(t_k) = 0$ for all impact (this true only during the 2-impact). The weak stability of the transition phase can be obtained by studying the variation of $V(X(t),t)$ between two impacts on the same surface.

t_{2k} represent instant of impact on $f_2(q) = 0$.

t_{2k+1} represent instant of impact on $f_1(q) = 0$.

$$q_{1d}^* = \begin{bmatrix} q_{1d}^{1*} \\ q_{1d}^{2*} \end{bmatrix} = \begin{bmatrix} -\alpha_1 V(X(\tau_0^k), \tau_0^k) \\ -\alpha_2 V(X(\tau_0^k), \tau_0^k) \end{bmatrix}$$

Let us calculate the following variation :

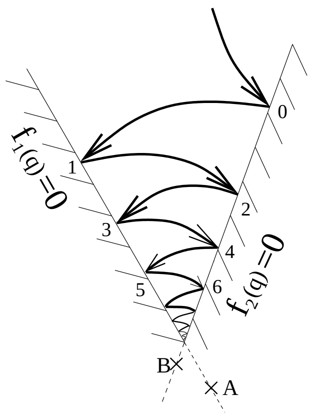

Figure 16. General case

$$
\begin{aligned}
&V(t^-_{2(k+1)}) - V(t^+_{2k}) \\
&= + \int_{t^+_{2k}}^{t^-_{2k+1}} \dot{V}(t)dt + \sigma_V(t_{2k+1}) \int_{t^+_{2k+1}}^{t^-_{2(k+1)}} \dot{V}(t)dt \\
&= \sigma_V(t_{2k+1}) - \gamma_2 \int_{t^+_{2k}}^{t^-_{2k+1}} \dot{q}^T \dot{q} dt - \gamma_2 \int_{t^+_{2k+1}}^{t^-_{2(k+1)}} \dot{q}^T \dot{q} dt \\
&\qquad + \gamma_1 q^{*}_{1d}{}^T [q_1]^{t_{2k+1}}_{t_{2k}} + \gamma_1 q^{*}_{1d}{}^T [q_1]^{t_{2(k+1)}}_{t_{2k+1}} \qquad (30) \\
&= \Delta + \gamma_1 q^{*}_{1d}{}^T (q_1(t_{2(k+1)}) - q_1(t_{2k})) \qquad (31) \\
&= \Delta + \gamma_1 q^{1*}_{1d}{}^T (q^1_1(t_{2(k+1)}) - q^1_1(t_{2k})) \qquad (32)
\end{aligned}
$$

where Δ is the sum of all negative terms in (30). (31) is deduced from (30) since $q^2_1(t_{2k}) = 0$ for all k.

With $\alpha_1 = 0$, we have $q^{1*}_{1d} = 0$ and then :

$$V(t^-_{2(k+1)}) - V(t^+_{2k}) < 0$$

The strategy is to take $\alpha_1 = 0$ (target A, see figure 16) at the begining of the transition phase to stabilize the system on $f_2(q) = 0$, and to switch to $\alpha_2 = 0,\ \alpha_1 > 0$ (target B, see figure 16) when the system is on $f_2(q) = 0$ (or to switch to the previous case).

5.2 Plastic impact if $\theta_{kinetic} > \frac{\pi}{2}$

The situation where $\theta_{kinetic} > \frac{\pi}{2}$ (see figure 17) is more complex. Indeed if $e_n > 0$, if $q(t^-_k) \in \partial\Phi_i$ we cannot be sure that $q(t^+_k) \in \partial\Phi_i$, and the work done above reasoning fail. Let us consider $e_n = 0$. In this case if $q(t^-_k) \in \Sigma_i$ then $q(t^+_k) \in \Sigma_j,\ j \neq i$. After the 2-impact the system is alway in contact with a surface, however not the same surface as before the 2-impact (see figure 17). The problem is to find a control law which stabilizes the system around the intersection of the two surfaces.

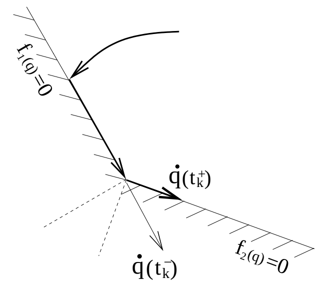

Figure 17. Generalized plastic 2-impacts

6 Conclusion

This paper deals with the tracking control of fully actuated Lagrangian systems subject to frictionless unilateral constraints. The aim of this paper is to study a controller for specific nonsmooth systems which perform cyclic impacting tasks. First the stability framework dedicated to study these systems is recalled, and some definitions are given. Then we precise the condition of existence of desired trajectories which give asymptotic stable controllers for this class of system.

The second part of this paper is devoted to numerically study an example : a 2-link plannar manipulator subject to a single unilateral constraint. This example allows to exhibit some results on the robustness of this control framework in term of uncertainty of the constraint surface position. The effect of measurement noise is also studied. It is shown that the proposed scheme possesses some interesting robustness properties.

The last part of this note is devoted to the case when so-called multiple impacts occur. Some specific difficulties related to the constraint boundary geometry, are highlighted, and some possible manners to extend the single constraint case are indicated.

Challenging goals are now to caracterise more precisely the subspace of initial conditions $\{CI\}$ which produce asymptotically strongly stable systems, and to extend the results on more general cases for multiple impacts.

References

Brogliato, B., Niculescu, S.I., & Orhant, P. 1997 (February). On the control of finite-dimensional mechanical systems with unilateral constraints. *Pages 200–215 of: IEEE Trans. on Automatic Control*, vol. 42.

Brogliato, B., Niculescu, S., & Monteiro-Marques, M. 2000. On tracking control of a class of complementary-slackness hybrid mechanical systems. *Systems & Control Letters*, 255–266.

Fremond, M. 2002. *NonSmooth Thermomechanics*. Springer.

Glocker, Ch. 2001. *Set Valued Force Laws: Dynamics of Non-Smooth Systems*. Lecture Notes in Applied Mechanics, vol. 1. Springer.

McClamroch, N.H., & Wang, D. 1988 (May). Feedback Stabilization and Tracking of Constrained Robots. *Pages 419–426 of: IEEE Trans. on automatic control*, vol. 33.

Moreau, J.J. 1988. Unilateral contact and dry friction in finite freedom dynamics *in Nonsmooth Mechanics and Applications*. CISM Courses and Lectures 302. Springer Verlag. 1-82.

Paden, B., & Panja, R. 1988. Globally asymptotically stable PD+ controller for robot manipulators. *Int. J. Control*, **47**, 1697–1712.

Van der Schaft, A., & Schumacher, J. M. 2000. *An Introduction to Hybrid Dynamical Systems*. Lecture Notes in Control and Information Sciences, no. 251. London: Springer.